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Semi-numerical simulation of groundwater flow induced by periodic forcing with a case-study at an island aquifer

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Least-squares fitting

Summary This paper presents an efficient method to analyze the aquifer response to periodic forcing. A complex transformation is used to change the time-dependent linear aquifer model into an equivalent time-independent problem of a complex function (referred to as the ''complex amplitude'' hereafter). The magnitude of the complex amplitude represents the amplitude of the sinusoidal fluctuation of the groundwater level, and the argument of the complex amplitude, the phase shift of the fluctuation. Different iterative methods can be used to obtain the numerical solution. A semi-analytical numerical solution of the groundwater level, which is numerical for the spatial variables and analytical for the time variable, can be obtained by implementing a simple inverse complex transform to the complex amplitude. The advantages of the time-independent method include ruling out of initial conditions, direct outputs of the amplitude and phase shift of different sinusoidal components of the water level fluctuations. On the contrary, the numerical solutions of the time-dependent model need further computational efforts for the analyses of the amplitudes and phase shifts of different sinusoidal components, and have to consider an initial condition whose impacts on the periodicity have to be eliminated by running sufficiently great number of tidal circles. A case study at an unconfined aquifer formed by fill materials near Lam Chau Island, Hong Kong SAR, PR China shows the applicability of the semi-numerical solution to analyze tide-induced watertable fluctuations. The aquifer parameters were estimated by minimizing a least-squares objective function to fit the complex amplitude of semidiurnal and diurnal components of the observed water level fluctuations. Compared with the parameter estimations of some of the previous analytical solutions, the numerical method gives not only better fitting to the water level observations, but also more aquifer parameters.

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Introduction

Strictly speaking, the groundwater flow in aquifers can be classified into three flow states: steady-state flow, non-periodic dynamic (or unsteady state) flow, and periodic flow, which are induced by steady-state, non-periodic, and periodic forcings, respectively. This paper focuses on the analysis of periodic groundwater flow induced by periodic forcing which changes sinusoidally in a linear aquifer system. The general periodic forcing which changes non-sinusoidally will not be addressed directly in this paper, because such a problem can be reduced to the sinusoidal problem by Fourier or other decomposition methods based on the superposition principle which is valid for linear aquifer system.

Periodic or approximately periodic forcing terms of an aquifer are commonplace in reality. For example, the sea tide or tidal stream is periodic forcing to the abutting aquifers, the seasonal precipitation and evapotranspiration that vary approximately periodically can be regarded as the periodic source term or boundary recharge (discharge) of an aquifer (Townley, 1995). It is important to distinguish the periodic flow from the non-periodic unsteady flow and the steady-state flow. Compared with the non-periodic unsteady flow, the periodic flow shows some steady properties: it repeats the same flow pattern periodically, and its model can be equivalently transformed into a time-independent one. Compared with the steady-state flow, the periodic flow shows some dynamic property: the aquifer's storativity plays an important role in characterizing periodic flow, but the steady-state flow is independent of the aquifer's storativity.

Since the 1950s, many analytical solutions have been derived to describe the periodic groundwater flow (e.g. Jacob, 1950; Nielsen, 1990; Jeng et al., 2002; Li et al., 2000; Li and Barry, 2000; Li et al., 2001; Townley, 1995; Sun, 1997; Jiao and Tang, 1999; Li and Jiao, 2001a,b; Li et al., 2002). Analytical studies are usually based on many strict assumptions. For example, the aquifer is assumed either to be isotropic, or to be homogeneous, or to have simple boundary and configurations. These assumptions restrict the applicability of the analytical solutions for the real aquifers. To relax these restrictions, numerical studies are necessary. A common method to obtain a numerical solution of the periodic aquifer model is to impose an extra hypothetical initial condition at a given initial time and then solve the corresponding discrete system of unsteady problem for every time step until the numerical solution asymptotically becomes periodic (e.g. Li et al., 2000; Volker and Zhang, 2001). The inadequacy of this method is that the numerical solution only provides the water level data. In order to obtain the amplitude and phase shift of the water level fluctuation, further data processing is needed. Moreover, in order to obtain the periodic solution independent of the hypothetical initial condition, the numerical solution has to be solved for many time steps. Consequently, this method is computationally demanding.

This paper presents an efficient semi-numerical method for periodic aquifer models by means of complex transformation. Based on this method, the time-dependent periodic aquifer model is transformed into an equivalent time-independent boundary-value problem of a complex elliptic equation with respect to an unknown complex function (referred to as the ''complex amplitude'' hereafter). The physical meanings of the complex amplitude are discussed and explained. The computational effort of the time-independent method for solving the complex amplitude is compared with that of the time-dependent one. The complex model of the complex amplitude is transformed into a real system of the real and imaginary parts of the complex amplitude, and an algorithm for obtaining the numerical solution of the real system is given. After the numerical solution of the complex amplitude is obtained, a semi-analytical numerical solution of the groundwater level, which is numerical for the spatial variables and analytical for the time variable, can be obtained immediately by implementing a simple inverse complex transform. Finally, a case study in an unconfined aquifer formed by fill materials near Lam Chau Island, Hong Kong SAR, PR China is used to examine the applicability of the semi-numerical solution to analyze tide-induced watertable fluctuations.

Mathematical model

Mathematical model for periodic groundwater flow

Consider the following general two-dimensional linear aquifer model (e.g. Bear, 1972)

$$S_S \frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(K_x \frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial H}{\partial y}\right), \quad -\infty < t < \infty, \quad (x, y) \in \Omega, \tag{1a}$$

where Ω is the aquifer domain considered, (x, y) is spatial coordinates and t is the time, $S_S(x, y)$ is the specific storage $[L^{-1}]$, $K_x(x, y)$ and $K_y(x, y)$ are the aquifer's hydraulic conductivity $[LT^{-1}]$ in the principal directions, which are assumed to be parallel to the x- and y-axes, respectively, and $H(x, y)$ is the hydraulic head [L] of the aquifer. The periodic forcing is implemented on part of the boundary $\partial\Omega_D$ of the domain Ω by the following Dirichlet boundary condition

$$H(x, y, t) = A(x, y)\cos(\omega t + \theta_D(x, y)), \quad -\infty < t < \infty, \quad (x, y) \in \partial\Omega_D, \tag{1b}$$

where $A(x, y)$ is the amplitude [L] of water level fluctuation on the boundary $\partial\Omega_D$, the constant ω is the frequency $[T^{-1}]$ of the sinusoidal fluctuation, $\theta_D(x, y)$ is the phase shift [radian] of the water level fluctuation on the boundary $\partial\Omega_D$. For coastal aquifers, Eq. (1b) describes the sinusoidal tidal boundary condition along the water-land interface (e.g. Sun, 1997; Li and Jiao, 2001a,b). On the remainder of the domain boundary $\partial\Omega_N$, a no-flow boundary condition is used, i.e.

$$n_x K_x \frac{\partial H}{\partial x} + n_y K_y \frac{\partial H}{\partial y} = 0, \quad -\infty < t < \infty, \quad (x, y) \in \partial\Omega_N, \tag{1c}$$

where (n_x, n_y) denotes the unit outward normal vector of the boundary.

In reality, the periodic forcing may appear not only in non-sinusoidal forms, but also in other terms, e.g. the boundary recharge in the Neumann boundary condition (1c) may be non-zero, or there may be an extra source term in the differential Eq. (1a), etc. Using the superposition principle, the solution corresponding to each individual forcing term can be considered separately. For the sake of succinctness, this paper will only consider the periodic forcing appearing in the Dirichlet boundary condition.

It is well-known that any non-sinusoidal forcing can be expressed as the sum of a steady term and a series of sinusoidal components. Based on the superposition principle, the aquifer's water level can also be expressed as the sum of a steady term and a series of solutions of the sinusoidal components. Therefore, the periodic Dirichlet boundary condition (1b) in the form of single sinusoidal component does not lose any generality.

Time-independent complex model

Model (1) is time-dependent. It can be simplified into a time-independent model by the following complex transformation

$$H(x,y,t) = \mathrm{Re}[Z(x,y)\exp(\mathrm{i}\omega t)], \tag{2}$$

where $\mathrm{i} = \sqrt{-1}$, ''Re'' denotes the real part of the followed complex expression, $Z(x,y)$ is a complex function of x and y, and is called complex amplitude due to its physical meaning that will be explained later. The dimension of the real and imaginary parts of $Z(x,y)$ is length (L). Substituting (2) into (1) results in three real equations. Using the equality

$$\begin{aligned} H(x,y,t) &= A(x,y)\cos(\omega t + \theta_D(x,y)) \\ &= \mathrm{Re}\{A\exp[\mathrm{i}(\omega t + \theta_D)]\}, \end{aligned} \tag{3}$$

and extending the above-mentioned three real equations into complex ones with respect to the complex function $Z(x,y)$, yield

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial Z}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial Z}{\partial y}\right) - \mathrm{i}\omega S_S Z = 0, \quad (x,y)\in\Omega, \tag{4a}$$

$$Z(x,y) = A(x,y)\exp(\mathrm{i}\theta_D(x,y)), \quad (x,y)\in\partial\Omega_D, \tag{4b}$$

$$n_x K_x\frac{\partial Z}{\partial x} + n_y K_y\frac{\partial Z}{\partial y} = 0, \quad (x,y)\in\partial\Omega_N. \tag{4c}$$

One can see that the time-independent model (4) is much simpler than the time-dependent model (1). If the complex amplitude $Z(x,y)$ is obtained by solving (4), the water level $H(x,y,t)$ can be immediately determined from Eq. (2) for any time and location. To analyze the relationship between the complex amplitude $Z(x,y)$ and the hydraulic head $H(x,y,t)$ more clearly, let

$$Z(x,y) = u(x,y) + \mathrm{i}v(x,y) = |Z(x,y)|\exp[\mathrm{i}\theta(x,y)], \tag{5a}$$

where $u(x,y)$ and $v(x,y)$ are the real and imaginary parts of $Z(x,y)$,

$$|Z(x,y)| = \sqrt{u^2(x,y) + v^2(x,y)} \tag{5b}$$

is the magnitude of $Z(x,y)$, $\theta(x,y)$ is the argument of $Z(x,y)$ defined as

$$\theta(x,y) = \arg(Z) = \begin{cases} \arctan\frac{v(x,y)}{u(x,y)}, & \text{if} \quad u>0, \quad v\geqslant 0, \\ \pi + \arctan\frac{v(x,y)}{u(x,y)}, & \text{if} \quad u\leqslant 0, \\ 2\pi + \arctan\frac{v(x,y)}{u(x,y)}, & \text{if} \quad u>0, \quad v<0. \end{cases} \tag{5c}$$

The range of $\theta(x,y)$ is $[0,2\pi)$. Substituting (5a) into (2), yields

$$\begin{aligned} H(x,y,t) &= \mathrm{Re}[Z(x,y)\exp(\mathrm{i}\omega t)] \\ &= |Z(x,y)|\cos(\omega t + \theta(x,y)). \end{aligned} \tag{6}$$

Eq. (6) gives the physical explanation of the complex amplitude $Z(x,y)$: Its magnitude $|Z(x,y)|$ denotes the amplitude of sinusoidal fluctuation of the groundwater level $H(x,y,t)$, its argument $\theta(x,y)$ denotes the phase shift of the sinusoidal fluctuation at the location (x,y). In particular, on the Dirichlet boundary $\partial\Omega_D$, $|Z(x,y)|$ and $\theta(x,y)$ become $A(x,y)$ and $\theta_D(x,y)$, respectively. Eq. (6) also implies that the water level $H(x,y,t)$ fluctuates in a sinusoidal way at any fixed location in the aquifer. Moreover, the complex amplitude $Z(x,y)$ directly gives the two key factors of the sinusoidally periodic fluctuation of the groundwater level $H(x,y,t)$: the amplitude and phase shift, hence it can describe the water level fluctuation succinctly and efficiently.

According to the Fredholm alternative property of the eigenvalue–eigenfunction theory on elliptic partial differential equations with complex coefficients (e.g. Chapter 3 of Mizhohada, 1973), the existence and uniqueness of solution to model (4) are based on the fact that in (4a), the real part of the coefficient of the complex amplitude $Z(x,y)$ is not positive. Another important fact to be emphasized is that models (1) and (4) are equivalent to each other due to the linear, non-singular complex transformation (2). Therefore, the existence and uniqueness of solution to model (1) are implied by those of model (4).

Numerical solution

Equivalent, time-independent real system of the complex model (4)

There are a few special cases when model (4) is simple enough to derive analytical solutions (e.g. Jacob, 1950; Townley, 1995; Sun, 1997; Jiao and Tang, 1999; Li et al., 2000; Li and Jiao, 2001a,b, 2002, Li et al., 2002). Generally, however, it is difficult or even impossible to pursue analytical solution to model (4). Hence a numerical solution is necessary. Because the coefficient matrix of the discrete governing equations of (4a) is complex, common, real matrix solvers (e.g. Golub and van Loan, 1983) are not suitable. If the discrete node number is small, the direct solution method such as complex LU decomposition of a complex matrix can be used to obtain the solution. In reality, however, the discrete node number is often much greater than 100. In this case, iterative solution methods are preferred due to the large, sparse data structure of the coefficient matrix. In addition to the LSQR iterative method (Paige and Saunders, 1982a,b), there are several other iterative methods such as DSLUCS (a Lanczos-type bi-conjugate gradient solver with incomplete LU-factorization as preconditioner) and DSLUGM (a generalized minimum residual solver) can be used to solve the discrete linear system of (4). Detailed information about DSLUCS, DSLUGM and other iterative methods can be found in Moridis and Pruess (1998). All these codes are, however, only for real systems. In order to use them, the complex problem (4) is equivalently changed into the following real system with respect to the real and imaginary parts, $u(x,y)$ and $v(x,y)$, of $Z(x,y)$

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial u}{\partial y}\right)+\omega S_S v=0,\quad (x,y)\in\Omega, \tag{7a}$$

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial v}{\partial y}\right)-\omega S_S u=0,\quad (x,y)\in\Omega, \tag{7b}$$

$$u=A(x,y)\cos\theta_D(x,y),\quad v=A(x,y)\sin\theta_D(x,y),\quad (x,y)\in\partial\Omega_D, \tag{7c}$$

$$n_xK_x\frac{\partial u}{\partial x}+n_yK_y\frac{\partial u}{\partial y}=n_xK_x\frac{\partial v}{\partial x}+n_yK_y\frac{\partial v}{\partial y}=0,\quad (x,y)\in\partial\Omega_N. \tag{7d}$$

Numerical solution of the time-independent model (7)

One can discretize the governing Eq. (7) using various methods such as finite-difference, finite-volume or finite-element method according to one's preference. In this paper the linear triangle finite element method is used. A similar approach was given by Smith et al. (2005) recently. Let $\{(x_i,y_i),\quad 1\leqslant i\leqslant N\}$ be the nodes of the triangulation, and $\phi_i(x,y)(1\leqslant i\leqslant N)$ be the linear basis function corresponding to the ith discrete node (x_i,y_i). Let

$$U=(u_1,\cdots,u_N)^T,\quad V=(v_1,\cdots,v_N)^T, \tag{8a}$$

where u_i and v_i denote the finite element approximations to $u(x_i,y_i)$ and $v(x_i,y_i)$, respectively, superscript ''T'' indicates the transpose of a vector or matrix. The corresponding complex vector

$$\bar{Z}=U+iV \tag{8b}$$

is called the complex amplitude vector. Based on the finite element method, $u(x,y)$ and $v(x,y)$ can be approximated by

$$u(x,y)\approx\sum_{j=1}^{N}u_i\phi_i(x,y),\qquad v(x,y)\approx\sum_{j=1}^{N}v_i\phi_i(x,y). \tag{9}$$

The discretized hydraulic conductivities $K_x(x,y)$, $K_y(x,y)$, and specific storage $S_S(x,y)$ are assumed to be constants on each triangle element $\mathbf{e}$, i.e.

$$K_x|_\mathbf{e}=K_x^e,\quad K_y|_\mathbf{e}=K_y^e,\quad S_S|_\mathbf{e}=S_S^e. \tag{10}$$

By applying the standard Galerkin finite element procedure (see, e.g. Pinder and Gray, 1977), the discrete governing equations of (7) can be written in matrix form as follows:

$$\begin{pmatrix}\mathbf{A} & -\omega\mathbf{S}\\ \omega\mathbf{S} & \mathbf{A}\end{pmatrix}\begin{bmatrix}U\\V\end{bmatrix}=\begin{bmatrix}F_u\\F_v\end{bmatrix}, \tag{11a}$$

where $\mathbf{A}$ is an $N\times N$ matrix with its (i,j) entry being

$$a_{ij}=\begin{cases}\delta_{ij}, & \text{if node } (x_i,y_i)\in\partial\Omega_D,\\ \sum_\mathbf{e}\int_\mathbf{e}\left(K_x^e\frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x}+K_y^e\frac{\partial\phi_i}{\partial y}\frac{\partial\phi_j}{\partial y}\right)dx\,dy, & \text{otherwise,}\end{cases} \tag{11b}$$

in which δ_{ij} equals 1 when $i=j$ and 0 when $i\neq j$, $\sum_\mathbf{e}$ means to sum the followed terms with respect to every triangle element $\mathbf{e}$ of the finite element triangulation; and $\mathbf{S}$ is an $N\times N$ matrix whose (i,j) entry is defined as

$$S_{ij}=\begin{cases}0, & \text{if node } (x_i,y_i)\in\partial\Omega_D,\\ \sum_\mathbf{e}\left(S_S^e\int_\mathbf{e}\phi_i\phi_j\,dx\,dy\right), & \text{otherwise}\end{cases} \tag{11c}$$

F_u and F_v are N-dimensional vectors determined by the boundary conditions in (7c), and their respective ith components, f_{ui} and f_{vi}, are defined as

$$f_{ui}=\begin{cases}A(x_i,y_i)\cos\theta_D(x_i,y_i), & \text{if node } (x_i,y_i)\in\partial\Omega_D,\\ 0, & \text{otherwise,}\end{cases} \tag{11d}$$

$$f_{vi}=\begin{cases}A(x_i,y_i)\sin\theta_D(x_i,y_i), & \text{if node } (x_i,y_i)\in\partial\Omega_D,\\ 0, & \text{otherwise.}\end{cases} \tag{11e}$$

One can show that the unsymmetric system (11a) has a unique solution. The details are given in Appendix A.

The above time-independent method based on the model (7) is better than the time-dependent method directly based on the model (1). The time-independent solution gives the two key factors – the amplitude and phase shift of the water level fluctuation for each point in the aquifer directly. These two factors characterize the local water level fluctuation more directly and clearly than the data of water level fluctuation with time. In the case that only the local water level fluctuation data are available, tedious comparison and calculation are needed to determine the amplitude and phase shift of the local fluctuation. Moreover, the time-independent method spares much calculation work compared with the time-dependent one. To obtain the numerical solution of the time-independent model (7), one needs to solve the unsymmetric system (11a) of order $2N$ only once; to obtain the numerical solution of the time-dependent model (1), however, one has to impose an extra hypothetical initial condition to model (1), and solve the discrete system of an unsteady-state problem for every time step until the numerical solution asymptotically becomes periodic.

Case study near Lam Chau Island, Hong Kong

Hydrogeological backgrounds

The 12.48 km^2 Chek Lap Kok airport platform was formed in January 1996 from excavation of the original islands of Chek Lap Kok and Lam Chau and reclamation of about three quarters of the area from the sea. Before reclamation, the sea bottom was predominantly very soft to soft marine mud. Under the mud was 8–16 m thick alluvial clay layer comprising the stiff clay, firm-to-stiff clay and stiff lenses which form the upper component of the Chek Lap Kok Formation. The mud was dredged prior to reclamation and replaced by fill materials, which consist of sands and rock boulders. The fill material becomes a very good unconfined aquifer. The alluvial clay can be regarded as impermeable bottom of the unconfined aquifer. Near Lam Chau Island, the dredge levels (or the elevation of the impermeable bottom of the unconfined aquifer) is −7.1 mPD (meter Principal Datum), the mean sea level is 1.4 mPD (Plant et al., 1998). So the aquifer's mean thickness b = 8.5 m. The sea-land boundary of the reclaimed land is formed by the fill materials and can be assumed approximately to be vertical.

All the original data in the following are taken from Provisional Airport Authority (1994a,b). The area reclaimed around Lam Chau Island in the middle of September 1994 is shown in Fig. 1. Two pits (Pits 2 and 3 in Fig. 1) were

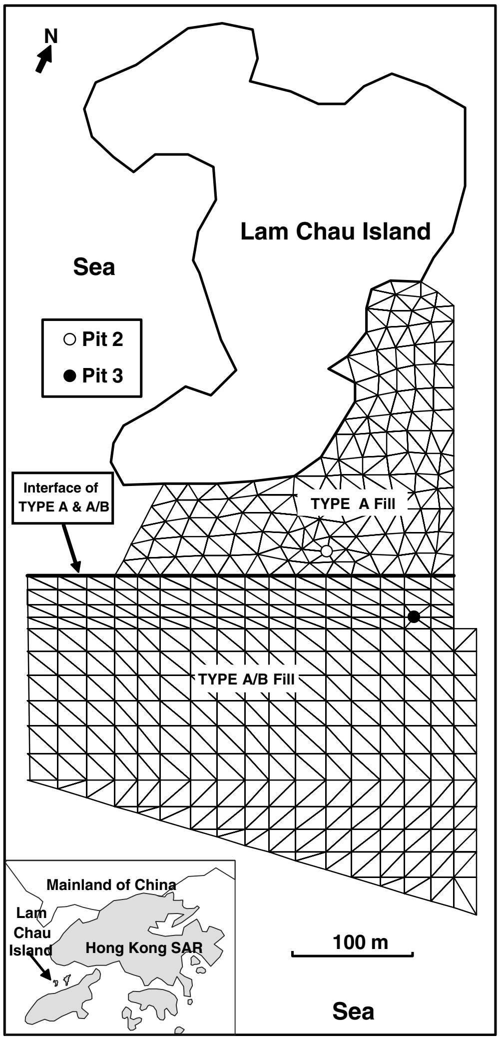

Figure 1 The area reclaimed around Lam Chau Island in the middle of September, 1994 and the finite element triangulation for the time-independent numerical solution.

excavated in the fill material to about 2–3 m below the watertable. There are two different kinds of fill materials: TYPE A and TYPE A/B. TYPE A fill comprises hard and durable rock excavated using drill and blast techniques. TYPE B fill comprises residual soil together with completely and highly weathered granite and corestones of fresher rock. TYPE A/B fill is a mixture of TYPE A and TYPE B fills (Plant et al., 1998). TYPE A fill abuts the southeastern boundary of Lam Chau Island immediately. TYPE A/B fill extends southward the aquifer formed by TYPE A fill. The interface of the two fill materials is located on the line $y = 0$ shown in Fig. 1. The unconfined aquifer formed by the two kinds of fill materials is surrounded by the sea from the west, east and south. The material of Lam Chau Island abutting the TYPE A fill is bedrock, so the interface between Lam Chau Island and TYPE A fill is regarded as impermeable.

Aquifer model setup

The sea-land boundary is assumed to be vertical. The sea tide near Lam Chau Island comprises semidiurnal and diurnal components. The amplitudes of both of the sinusoidal components are less than 0.8 m (see Table 1) and much less than the mean thickness ($b = 8.5$ m) of the unconfined aquifer. Therefore, the Boussinesq's equation for describing watertable variations in an unconfined aquifer can be linearized (see e.g. Bear, 1972) into Eq. (1a) to describe the watertable in the unconfined aquifer, and the over-height effect of watertable induced by the sea tide can be neglected safely. In this case, the domain Ω in Eq. (1a) is composed of TYPE A and TYPE A/B fills shown in Fig. 1; the symbol $H(x, y, t)$ represents the watertable of the unconfined aquifer. Because there is no detailed information about the heterogeneity of the two fill materials, averaged values of the permeability and specific yield of each of the fill materials are used in the model. The symbol $S_S(x, y)$ in Eq. (1a) represents S_A/b in the TYPE A fill area and $S_{A/B}/b$ in the TYPE A/B fill area with S_A and $S_{A/B}$ being the specific yields of TYPE A and TYPE A/B fills, respectively; the aquifer is assumed to be isotropic,

Table 1 Fitting results of tidal level and watertables at Pits 2 and 3

	h_0 (m)	A_1 (m)	θ_1 (Radian)	A_2 (m)	θ_2 (Radian)	Mean error (m)	Standard Deviation (m)
Direct least-squares fitting of water table at Pit 2	1.40	0.356	0.7123	0.346	−2.0424	−0.00403	0.03139
Fitting at Pit 2 by model (7)	Inapplicable	0.362	0.7082	0.346	−2.0424	0.00586	0.03210
Fitting at Pit 2 of analytical solution by Townley (1995)	Inapplicable	0.4167	0.5776	0.3969	−2.0102	0.00661	0.07413
Least-squares fitting of tidal level when measuring Pit 2	1.42	0.576	1.4916	0.441	−1.4813	−0.0001	0.02688
Direct least-squares fitting of watertable at Pit 3	1.37	0.471	−0.0786	0.1247	0.3307	−0.0053	0.0169
Fitting at Pit 3 by model (7)	Inapplicable	0.4673	−0.0683	0.1246	0.3307	−0.00594	0.0173
Fitting at Pit 3 of analytical solution by Townley (1995)	Inapplicable	0.4048	−1.3345	0.1012	0.4722	0.03149	0.3365
Least-squares fitting of tidal level when measuring Pit 3	1.39	0.7715	−0.6894	0.1623	0.9267	−0.00202	0.0362

so $K_x(x,y) = K_y(x,y) = K(x,y)$, which equals constant K_A in the TYPE A fill area and $K_{A/B}$ in the TYPE A/B fill area. As a result, the model (7a) and (7b) actually have only three independent parameters: K_A/S_A, $K_{A/B}/S_{A/B}$ and $S_A/S_{A/B}$.

The watertables at Pits 2 and 3 were observed for about 24 h in different time periods of September, 1994 and are shown in dots in Figs. 2 and 3, respectively. The data of sea tidal level in the same periods at Lam Chau Island are shown in circles in Figs. 2 and 3, respectively. The fluctuations of the sea level along the sea-land boundary $\partial\Omega_D$ and the watertable in Pits 2 and 3 are approximated by the superposition of the semidiurnal ($\omega_1 = 0.51\ h^{-1}$) and diurnal ($\omega_2 = 0.26\ h^{-1}$) components, i.e.

$$H(x,y,t) = h_0(x,y) + A_1(x,y)\cos(\omega_1 t + \theta_1(x,y)) + A_2(x,y)\cos(\omega_2 t + \theta_2(x,y)), \quad (12)$$

where h_0, A_j, ω_j and θ_j ($j = 1,2$) are the mean elevation [L], amplitude [L], frequency [T^{-1}], and the phase shift [radian] of the water level fluctuations, respectively, and the subscript $j = 1,2$ represents the semidiurnal and diurnal components, respectively. Their values are calculated using least-squares fitting to the observed data and listed in Table 1. The fitting curves given by (12) are shown in Figs. 2 and 3.

Along the interface of Lam Chau Island and TYPE A fill, no-flow boundary condition (1c) is used.

Neglecting the steady component of the watertable $H(x,y,t)$, the following decomposition can be made

$$H(x,y,t) = H_1(x,y,t) + H_2(x,y,t), \quad (13a)$$

where $H_1(x,y,t)$ and $H_2(x,y,t)$ are the semidiurnal and diurnal sinusoidal components. Using the boundary values of A, ω and θ_D determined by the least-squares fitting and listed in Table 1, the numerical solutions of (7) are calculated for the semidiurnal and diurnal components, respectively, to obtain the semidiurnal complex amplitude $Z_1 = u_1 + iv_1$ and the diurnal complex amplitude $Z_2 = u_2 + iv_2$. Finally, based on (5), (6) and (13a), the watertable in the unconfined aquifer is given by the semi-analytical numerical solution

$$H(x,y,t) = \sum_{j=1}^{2} H_j(x,y,t) = \sum_{j=1}^{2} |Z_j|\cos(\omega_j t + \arg Z_j), \quad (13b)$$

which is numerical for the spatial variables (x,y) and analytical to the time variable t.

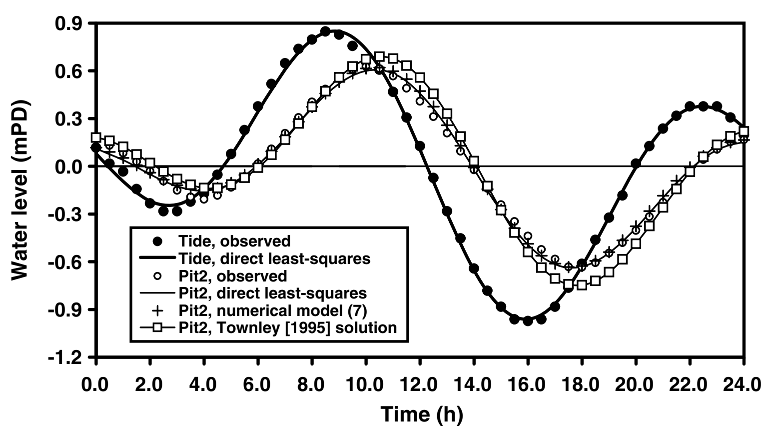

Figure 2 Observed sea level and watertable at Pit 2 over a period of about 24 h in September of 1994 as well as their fittings by different methods.

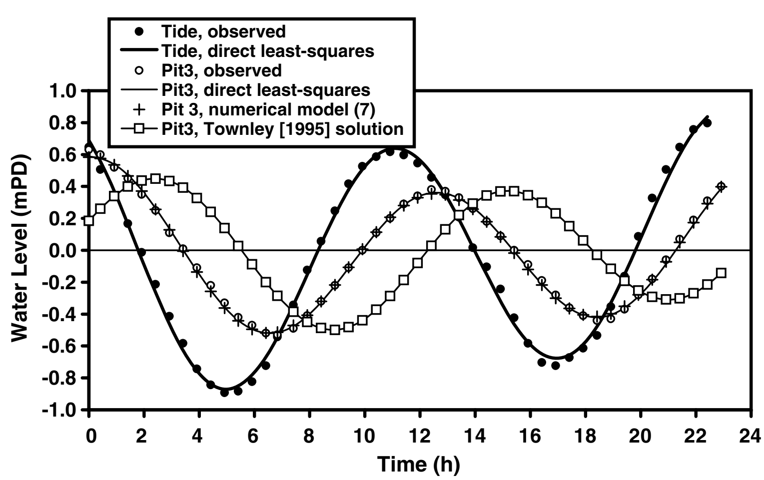

Figure 3 Observed sea level and watertable at Pit 3 over a period of about 24 h in September of 1994 as well as their fittings by different methods.

Aquifer parameter identification and discussions

In order to estimate the three independent aquifer parameters K_A/S_A, $K_{A/B}/S_{A/B}$ and $S_A/S_{A/B}$, the following minimizing problem

$$\min_{K_A/S_A,\ K_{A/B}/S_{A/B},\ S_A/S_{A/B}} \sum_{k=2}^{3}\sum_{j=1}^{2}\left[(u_j - A_j\cos\theta_j)^2|_{\text{Pit}k} + (v_j - A_j\sin\theta_j)^2|_{\text{Pit}k}\right] \quad (14)$$

is solved using Gauss-Newton method (Yeh, 1986), where the values of A_j and θ_j (j = 1,2) were determined by least-squares fitting to the observed watertable at Pits 2 and 3, respectively (see Table 1), by minimizing the least-squares objective functions using the method by Shanno and Phua (1976, 1980); u_j and v_j are numerical solutions of the model (7) at Pits 2 and 3, and dependent on the aquifer parameters K_A/S_A, $K_{A/B}/S_A$ and $S_A/S_{A/B}$. The objective function contains three unknown parameters and eight squared terms derived from the real and imaginary parts of the complex amplitude, 2 pits and 2 sinusoidal components of the sea tide. Generally speaking, the more the least-squares terms and the less the parameters numbers in an objective function, the more stable the estimation result of the unknown parameters. So the results of the minimizing problem (14) have adequate stability. The values of estimated aquifer parameter are shown in the last row of Table 2.

To determine the initial values of the three parameters when minimizing (14), the parameters K_A/S_A and $K_{A/B}/S_{A/B}$ were roughly estimated using Jacob (1950) solution by fitting the amplitudes and phase shifts of the semidiurnal and diurnal components of the watertables at Pits 2 and 3 (see Table 2). The initial guess of $S_A/S_{A/B}$ is chosen to be 2.0. It was found that all the minimizing processes starting from different initial values given in Table 2 converged to the same results. Jacob (1950) solution cannot be used to estimate the specific yield ratio $S_A/S_{A/B}$ while the semi-analytical method can. Moreover, the parameters estimated by Jacob (1950) solution vary in a relatively wide range and most of them differ significantly from those obtained by the semi-analytical method.

Figs. 2 and 3 show the observed and fitted watertables at Pits 2 and 3, respectively. The curves fitted by the semi-analytical model almost coincide with the least-squares fittings to the observations. In order to examine the impacts of the irregular boundary shape and the different hydrological properties of the two fill materials on the tide-induced watertable fluctuations, the analytical solution of Townley (1995) for describing tidal fluctuations in an island with a constant width was used to predict the watertable in Pits 2 and 3 using the parameter values K_A/S_A = 512.3 m/h and $K_{A/B}/S_{A/B}$ = 79.00 m/h identified by our method. The width was approximated by a value of 282 m for Pit 2 and 364 m for Pit 3. One can see that the curves predicted by Townley (1995) solution deviate significantly from the observations, indicating the significant impacts of the irregular boundary shape and different hydrological properties of the two fill materials on the tide-induced watertable fluctuations.

Table 1 gives values of fluctuation amplitudes and phase shifts of semidiurnal and diurnal components of different fittings to the observed water level fluctuations in the sea and at Pits 2, 3. The last two columns of Table 1 are the mean error and the standard deviation of the fittings. The mean error is defined as $\text{ME} = \frac{1}{n}\sum_{i=1}^{n}[h_{\text{fitted}}(t_i) - h_{\text{observed}}(t_i)]$ and the standard deviation is defined as $\{\frac{1}{n}\sum_{i=1}^{n}[h_{\text{fitted}}(t_i) - h_{\text{observed}}(t_i) - \text{ME}]^2\}^{1/2}$, where n is the data number during the observation period, $h_{\text{observed}}(t_i)$ is the observed watertable or sea level at time t_i, while $h_{\text{fitted}}(t_i)$ is its fitting value. One can see that all the mean errors are very small due to the periodicity of the observed data. The standard deviations vary in a very narrow range from 0.017 to 0.034 m not only for the direct least-squares fittings of the sea level and watertables at Pits 2 and 3, but also for the fittings based on model (7) at Pits 2 and 3. For fittings based on the analytical solution of Townley (1995), the standard deviation at Pits 2 and 3 equals 0.074 and 0.337 m, respectively, which are significantly greater than the others. This indicates that our model cannot be simplified as a homogeneous island with constant width. Moreover, Townley (1995) solution cannot be used to estimate the specific yield ratio $S_A/S_{A/B}$.

There are four unknown parameters in the model. Because there is no sink or source term in the governing differential equation, it is theoretically impossible to obtain the estimations of all the four parameters. In spite of this, as shown above, our method can determine three of the four parameters from the observation data of only 2 points (Pits 2 and 3). In fact, the watertable fluctuation data at each pit include both diurnal and semidiurnal components, and in addition, each component can provide controls both of its amplitude and phase shift. Therefore, two observation points are adequate for the parameter estimations. If one knows any one of four parameters, then our method will give the estimations of all four parameters. This is to say, our method reduces the unknown parameters from 4 to 1 for the case study.

Table 2 The estimated aquifer parameter values based on different method (N/A means that data are not available)

	K_A/S_A (m/h)	$K_{A/B}/S_{A/B}$ (m/h)	$S_A/S_{A/B}$ (–)
Jacob (1950) solution, amplitude fitting of semidiurnal component at Pit 2	1520	N/A	N/A
Jacob (1950) solution, amplitude fitting of diurnal component at Pit 2	3048	N/A	N/A
Jacob (1950) solution, phase shift fitting of semidiurnal component at Pit 2	579	N/A	N/A
Jacob (1950) solution, phase shift fitting of diurnal component at Pit 2	570	N/A	N/A
Jacob (1950) solution, amplitude fitting of semidiurnal component at Pit 3	N/A	135.7	N/A
Jacob (1950) solution, amplitude fitting of diurnal component at Pit 3	N/A	242.7	N/A
Jacob (1950) solution, phase shift fitting of semidiurnal component at Pit 3	N/A	88.63	N/A
Jacob (1950) solution, phase shift fitting of diurnal component at Pit 3	N/A	47.46	N/A
Method proposed in this paper	512.3	79.00	1.898

Summary

An efficient method based on the complex transformation is proposed to analyze the aquifer response to periodic forcing. Using the method of Fourier series, any periodic non-sinusoidal forcing can be expressed as the sum of a steady term and a series of sinusoidal components. According to the superposition principle, the aquifer's water level can be expressed as the sum of the solutions to the decomposition terms of the periodic non-sinusoidal forcing. Hence, the use of a simple boundary condition containing only one single sinusoidal component does not lose any generality. Based on this method, the time-dependent linear aquifer model is transformed into equivalent time-independent problem with respect to the unknown complex amplitude. The magnitude of the unknown complex amplitude represents the amplitude of the sinusoidal fluctuation of the groundwater level, and the argument of the complex amplitude, the phase shift of the fluctuation. Therefore, the complex amplitude solution gives directly the two key factors – the amplitude and phase shift of the water level fluctuation for each point in the aquifer. These two factors characterize the local water level fluctuation more directly and clearly than the asymptotic numerical solution of the time-dependent model. These facts also imply that, for a linear aquifer model, the groundwater level will be sinusoidal if the forcing such as boundary conditions and source terms (e.g. precipitation, recharge, etc.) are sinusoidal, even if the aquifer is heterogeneous and anisotropic. The complex amplitude can be calculated much more easily than the time-dependent groundwater level. To obtain the numerical solution of the time-independent model, one needs to solve the discrete system only once. On the other hand, to obtain the numerical solution of the time-dependent model, one has to solve the corresponding discrete system for every time step until the numerical solution asymptotically becomes periodic. Moreover, the analyses of the amplitudes and phase shifts of different sinusoidal components need further computational efforts for the numerical solutions of the time-dependent model.

The semi-analytical numerical solution, which is numerical for the spatial variables and analytical for the time variable, can be obtained by implementing a simple inverse complex transform to the numerical solution of the complex amplitude. To avoid calculations of complex numbers in the numerical solution of the complex amplitude, the complex differential equation is transformed into an equivalent real system comprising two real governing differential equations of the real and imaginary parts of the complex amplitude. A case study in an unconfined aquifer formed by fill materials near Lam Chau Island, Hong Kong SAR, PR China shows the applicability of the semi-numerical solution to analyze tide-induced watertable fluctuations. The specific yield and permeability values of a piecewise-homogeneous aquifer were estimated by minimizing a least-squares objective function to fit the complex amplitude of semidiurnal and diurnal components of the observed water level fluctuations. Compared with the parameter estimations of the Jacob (1950) and Townley (1995) analytical solutions, the numerical method gives not only better fitting to the water level observations, but also more aquifer parameters.

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Appendix A. Existence and uniqueness of solution to system (11)

Let

$$F = F_u + \mathrm{i}F_v \tag{A1}$$

be an n-dimensional complex vector, then the real system (11a) can be equivalently written in complex form as

$$(\mathbf{A} + \mathrm{i}\omega\mathbf{S})\bar{Z} = F \tag{A2}$$

with $\bar{Z}$ defined by (8b). According to the Dirichlet boundary condition (or directly from (11b)–(11e)), for a node $i_0 \in \partial\Omega_\mathrm{D}$, one can see that the i_0th component z_{i_0} of $\bar{Z}$ is known and equals $A(x_{i_0}, y_{i_0})\exp[c(x_{i_0}, y_{i_0})]$, and that the off-diagonal elements of the i_0th row of the complex matrix $\mathbf{A} + \mathrm{i}\omega\mathbf{S}$ are zero. Therefore, all the terms containing z_{i_0} in the equations other than the i_0th equation of (A2) can be moved to the right hand side so that all the off-diagonal elements of its i_0th column of $\mathbf{A} + \mathrm{i}\omega\mathbf{S}$ become zero. After this procedure has been done for all the nodes on the Dirichlet boundary $\partial\Omega_\mathrm{D}$, the system (A2) is changed into

$$(\mathbf{A}_0 + \mathrm{i}\omega\mathbf{S}_0)\bar{Z} = F_0, \tag{A3}$$

where $\mathbf{A}_0$ is an $n \times n$ matrix whose (i, j) entry equals δ_{ij} if either the ith node or the jth one are on the Dirichlet boundary $\partial\Omega_\mathrm{D}$, otherwise, it equals that of $\mathbf{A}$; $\mathbf{S}_0$ is an $n \times n$ matrix whose (i, j) entry equals 0 if either the ith node or the jth one are on the Dirichlet boundary $\partial\Omega_\mathrm{D}$, otherwise, it equals that of $\mathbf{S}$; and the jth component of F_0 is given by

$$f_{0i} = \begin{cases} A(x_i, y_i)\exp[\mathrm{i}\theta_\mathrm{D}(x_i, y_i)] & \text{if node } i \in \partial\Omega_\mathrm{D}, \\ f_{ui} + \mathrm{i}f_{vi} + \sum\limits_{j \in \partial\Omega_\mathrm{D}} (a_{ji} + \mathrm{i}s_{ji})A(x_j, y_j)\exp[\mathrm{i}\theta_\mathrm{D}(x_j, y_j)], & \text{otherwise.} \end{cases} \tag{A4}$$

According to the definitions of $\mathbf{A}_0$ and $\mathbf{S}_0$ and (11b),(11c), one can easily conclude that both $\mathbf{A}_0$ and $\mathbf{S}_0$ are symmetric. Moreover, $\mathbf{A}_0$ is positive definite because it is the coefficient matrix of the discrete system of (4) when the imaginary term $\mathrm{i}\omega S_S Z$ vanishes, as is equivalent to the usual steady-state aquifer model whose discrete system has a positive definite coefficient matrix (e.g. Langtangen, 1999). Therefore, there exists a non-singular matrix $\mathbf{P}$, such that $\mathbf{A}_0 = \mathbf{P}\mathbf{P}^\mathrm{T}$. Using this, it follows that

$$\mathbf{A}_0 + \mathrm{i}\omega\mathbf{S}_0 = \mathbf{P}(\mathbf{I}_n + \mathrm{i}\omega\mathbf{P}^{-1}\mathbf{S}_0(\mathbf{P}^\mathrm{T})^{-1})\mathbf{P}^\mathrm{T}, \tag{A5}$$

where $\mathbf{I}_n$ is the identity matrix of order n. Because $\mathbf{P}^{-1}\mathbf{S}_0(\mathbf{P}^\mathrm{T})^{-1}$ is a real, symmetric matrix, all its eigenvalues are real, so the real part of any eigenvalue of the matrix $\mathbf{I}_n + \mathrm{i}\omega\mathbf{P}^{-1}\mathbf{S}_0(\mathbf{P}^\mathrm{T})^{-1}$ equals 1 and consequently $\mathbf{I}_n + \mathrm{i}\omega\mathbf{P}^{-1}\mathbf{S}_0(\mathbf{P}^\mathrm{T})^{-1}$ is non-singular. Finally, $\mathbf{A}_0 + \mathrm{i}\omega\mathbf{S}_0$ is non-singular according to (A5).

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