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Influence of the tide on the mean watertable in an unconfined, anisotropic, inhomogeneous coastal aquifer

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Abstract

For an isotropic, homogeneous coastal aquifer, previous studies [Aust J Phys 26 (1973) 513; Water Resour Res 17 (4) (1981) 1222] have found that the mean watertable is influenced by the sea tide and will stand considerably above the mean sea level even in the absence of net inland recharge. This paper investigates the influence of the tide on the mean watertable for the case that the unconfined coastal aquifer is inhomogeneous and anisotropic. A two-dimensional free surface model is considered under the following assumptions: (a) the principle directions of the hydraulic conductivities are horizontal and vertical; (b) the horizontal hydraulic conductivity $K_x(y)$ depends on the depth of the aquifer only and the vertical hydraulic conductivity $K_y(x, y)$ is arbitrary; (c) the water–land boundary is vertical and there is no seepage face; (d) the specific storage is constant wherever the free surfaces are located; and (e) there is no net inland recharge. An integral equation satisfied by the asymptotic watertable as the landward distance approaches infinity is derived for multi-sinusoidal-component sea tide. This equation suggests that the asymptotic watertable is independent of $K_y(x, y)$ and higher than mean sea level for arbitrary $K_x(y)$. The asymptotic watertable is solved explicitly from the integral equation for three common patterns of $K_x(y)$: constant, piecewise constant and linearly decreasing with the depth. Compared with the first two patterns, the third pattern has significant enhancing effect on the asymptotic watertable. Several previously published analytical solutions and the experiment data are in line with the analytical solution of this paper.

1. Introduction

The influence of the tidal water level fluctuation on the mean watertable in coastal aquifers is an interesting topic for hydrologists. For example, Philip [7], Smiles and Stokes [10], Knight [2], Parlange et al. [6] and Nielsen [5] studied the influences of the sea tide on the mean watertable in a coastal unconfined aquifer using different methods. Barry et al. [1] considered the effects of capillarity on the periodic solutions of the shallow flow approximation.

Philip [7] considered the mean watertable in an unconfined, isotropic and homogeneous coastal aquifer bounded by an impermeable bottom and a straight coastline with vertical beach connected to a sinusoidal tidal water body. He concluded that, even in the absence of net inland recharge of groundwater and rainfall, the difference between the asymptotic mean watertable as the landward distance approaches infinity and the mean sea level is given by

$$\Delta = b\left(\sqrt{1+\frac{1}{2}\frac{A^2}{b^2}}-1\right) \tag{1}$$

where A [L] is the tidal amplitude, and b [L] is the depth of the unconfined aquifer below the mean sea level. Philip's result was derived from the Boussinesq equation, which is based on the Dupuit–Forchheimer (D–F) assumptions that assume groundwater flow to be essentially horizontal. Knight [2] considered a free surface problem that took the vertical flow into account. He proved theoretically that Philip's result exactly holds independent of the validity of the D–F assumptions for an isotropic, homogeneous aquifer. Philip's theoretical prediction was examined and confirmed by a Hele–Shaw experiment conducted by Smiles and Stokes [10]. Parlange et al. [6] used second-order theory to describe the propagation of steady periodic motion of liquid in a

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porous medium. Their two laboratory experiments, together with analytical and numerical analyses also support Philip's [7] prediction. Nielsen [5] developed an approximate analytical solution based on a perturbation method to investigate the mean watertable in the inland region near the coastline. One of the limitations of all these studies is that they assume the sea tide has only one sinusoidal component. In reality, the sea tide consists of tens of sinusoidal components that include the effects of the sun, moon and earth, etc. [4,8]. Due to nonlinearities of the model equations describing the unconfined aquifer, the superposition principle does not apply and the consideration of all the tidal components is necessary. Also based on the Boussinesq equation, Li and Jiao [3] considered the tide-influenced mean water levels in a multi-layered leaky coastal aquifer system consisting of an unconfined aquifer, a confined aquifer and a semipermeable layer between them. They included different sinusoidal components of the sea tide in the tidal boundary condition. When the middle layer becomes impermeable and the sea tide has only one sinusoidal component, their solution in the unconfined aquifer collapses into those of Phillip [7] and Nielsen [5]. All the above-mentioned theoretical and numerical results use the assumptions of isotropy and homogeneity of the aquifer, and some of them are based on the D–F assumptions. These assumptions have significant discrepancy for real aquifers. For example, the real aquifer is usually anisotropic and inhomogeneous. Moreover, there is significant vertical flow in coastal aquifers near the coastline so the D–F assumptions are not valid. It is interesting to consider the influence of the sea tide on the mean water levels in a more realistic case. Based on such motivations, this paper investigates the influence of the tidal water level fluctuation on the asymptotic watertable as the landward distance approaches infinity in an unconfined, anisotropic, inhomogeneous coastal aquifer. A nonlinear free surface mathematical model is used. Different sinusoidal components of the sea tide are included in the tidal boundary condition. The behavior of the asymptotic watertable far from the coastline is analyzed. Explicit expressions of the differences between the asymptotic watertable and the mean sea level are obtained for common place aquifers such as a multi-layered aquifer, and an aquifer whose horizontal hydraulic conductivity decreases linearly with depth. The results are discussed and compared with existing results. In order to derive the asymptotic analytical solution, the density difference between the seawater and groundwater is ignored and the beach is assumed to be vertical although it is believed that a sloping beach of the tidal water has a significant effect on the mean watertable [5]. Therefore, this paper only gives a conservative estimation to the effect of the sea tide on enhancing the mean watertable for a real unconfined coastal aquifer with sloping beach.

2. Mathematical model and definitions of mean water levels

2.1. Mathematical model

Consider an unconfined coastal aquifer with a horizontal impermeable bottom and a vertical boundary with the tidal water. Cartisian coordinates x, y will be used, with the x-axis taken to be the bottom line of the aquifer and positive landward, and y to be positive upward (see Fig. 1). The aquifer extends landward without bound. As an analog to the real situation, the aquifer is assumed to be anisotropic with its two principle directions being horizontal and vertical, respectively. The capillary fringe [1] is neglected and it is assumed that there is no seepage face so that there exists a well-defined free surface $y = \eta(x,t)$ between the saturated and unsaturated zones. Groundwater flow below the free surface is driven by the piezometric head $\phi(x,y,t)$. Note that in a free surface problem the volume of water released from storage due to the compressibility of the aquifer can be neglected when compared with the volume of water released from the storage as the watertable is displaced. Consequently, the specific storage can be set to zero and $\phi(x,y,t)$ satisfies the two-dimensional elliptic equation [9]

$$K_x(y)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial}{\partial y}\left(K_y(x,y)\frac{\partial\phi}{\partial y}\right) = 0, \tag{2a}$$

where $K_x(y)$ and $K_y(x,y)$ are the horizontal and vertical hydraulic conductivities [LT^{-1}] of the aquifer, respectively. It is assumed that $K_x(y)$ depends on the vertical location y only. If the aquifer is composed of different horizontal layers, the vertical hydraulic conductivity $K_y(x,y)$ will change discontinuously at the interface of two adjacent layers. In this case the piezometric head $\phi(x,y,t)$ satisfies the partial differential equation (2a) inside each of the layers, and at each interface of two layers where $y = y_j$ ($j = 1,\ldots,L-1$, L is the number of layers), it satisfies the head continuity and flux continuity conditions

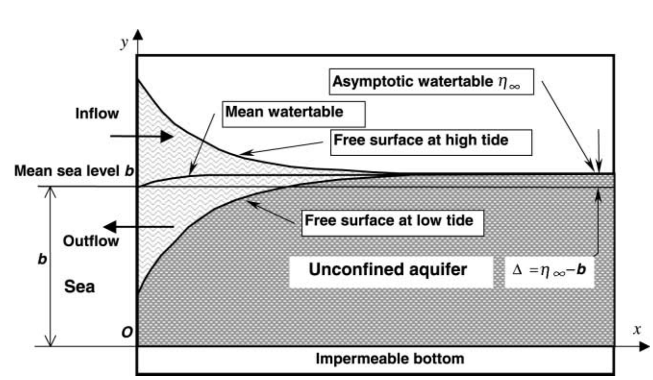

Fig. 1. Schematic of an unconfined aquifer near open tidal water.

$$\lim_{y\downarrow y_j} \phi(x,y,t) = \lim_{y\uparrow y_j} \phi(x,y,t),$$
$$\lim_{y\downarrow y_j} K_y(x,y)\frac{\partial \phi}{\partial y} = \lim_{y\uparrow y_j} K_y(x,y)\frac{\partial \phi}{\partial y}. \tag{2b}$$

The tidal boundary condition at the water–land interface along the coastline is given by

$$\phi(x,y,t)|_{x=0} = W_{\mathrm{Tide}}(t) \stackrel{\mathrm{def.}}{=} b + \sum_{j=1}^{N} A_j \cos(\omega_j t + c_j), \tag{2c}$$

where $W_{\mathrm{Tide}}(t)$ is the tidal water level [L]; b is the depth of the aquifer below the mean sea level and equals the mean sea level when the datum of the model is set to be the bottom of the aquifer (see Fig. 1, b is assumed to be great enough so that the tidal water remains connected with the aquifer at low tide), N is the number of the sinusoidal components of the tide in a specified coastal area; A_j, ω_j and c_j $(j = 1, \dots, N)$ are the amplitude [L], angular velocity [T^{-1}] and phase shift of the jth sinusoidal component of the tide, respectively. The angular velocities $\omega_1, \dots, \omega_N$ are not equal to each other. The bottom of the aquifer is impermeable so a no-flow boundary condition

$$\left.\frac{\partial \phi}{\partial y}\right|_{y=0} = 0 \tag{2d}$$

is used. As x approaches infinity, it is assumed that there is no inland recharge, i.e.,

$$\lim_{x\to\infty} \frac{\partial \phi}{\partial x} = 0. \tag{2e}$$

On the free surface, the water is in contact with air at atmosphere pressure. Therefore, one has a dynamic boundary condition

$$\phi(x,y,t)|_{y=\eta(x,t)} = \eta(x,t). \tag{2f}$$

On the other hand, the free surface is defined by the property that the groundwater flow does not cross it. Hence the velocity of the groundwater flow normal to the free surface must be equal to the velocity of the free surface normal to itself (e.g. Whitham [11]). Based on this and the no-recharge assumption, one obtains the kinematic boundary condition of the free surface

$$K_x(y)\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} - K_y(x,y)\frac{\partial \phi}{\partial y} = S_y \frac{\partial \eta}{\partial t}, \quad \text{if } y = \eta(x,t), \tag{2g}$$

where S_y is the specific yield and assumed to be a constant.

2.2. Integral simplification of the free surface model

Define the quantity $W(x,t)$ as

$$W(x,t) = \int_0^{\eta(x,t)} K_x(y)[\phi(x,y,t) - y]\,\mathrm{d}y. \tag{3}$$

For a given horizontal hydraulic conductivity $K_x(y)$, the integral may be computed explicitly at the boundary $x = 0$, where the pressure distribution is known, as

$$W(0,t) = \int_0^{\eta(0,t)} K_x(y)[\phi(0,y,t) - y]\,\mathrm{d}y$$
$$= \int_0^{W_{\mathrm{Tide}}(t)} K_x(y)[W_{\mathrm{Tide}}(t) - y]\,\mathrm{d}y. \tag{4}$$

Here the assumption that there is no seepage-face is used. Based on (3) and (2f), by making use of the Leibnitz rule for differentiation under the integral sign, the total instantaneous flux $Q(x,t)$ through a vertical plane perpendicular to the x axis can be written as

$$\frac{\partial W}{\partial x} = \int_0^{\eta(x,t)} K_x(y)\frac{\partial \phi}{\partial x}\,\mathrm{d}y + \left.\left\{K_x(y)[\phi(x,y,t) - y]\frac{\partial \eta}{\partial x}\right\}\right|_{y=\eta(x,t)}$$
$$= \int_0^{\eta} K_x(y)\frac{\partial \phi}{\partial x}\,\mathrm{d}y = -Q(x,t). \tag{5}$$

When x approaches infinity, using (5) and (2e), it follows that

$$\lim_{x\to\infty} \frac{\partial W}{\partial x} = \lim_{x\to\infty} \int_0^{\eta} K_x(y)\frac{\partial \phi}{\partial x}\,\mathrm{d}y = 0. \tag{6}$$

Differentiating (5) with respect to x once more, and using (2a), (2b), yield

$$\frac{\partial^2 W}{\partial x^2} = \frac{\partial}{\partial x}\left[\int_0^{\eta} K_x(y)\frac{\partial \phi}{\partial x} dy\right]$$
$$= \frac{\partial}{\partial x}\left[\sum_{j=1}^{L}\int_{y_{j-1}}^{y_j} K_x(y)\frac{\partial \phi}{\partial x}\,\mathrm{d}y\right]$$
$$= K_x(y)\frac{\partial \eta}{\partial x}\left.\frac{\partial \phi}{\partial x}\right|_{y=\eta} + \sum_{j=1}^{L}\int_{y_{j-1}}^{y_j} K_x(y)\frac{\partial^2 \phi}{\partial x^2}\,\mathrm{d}y$$
$$= K_x(y)\frac{\partial \eta}{\partial x}\left.\frac{\partial \phi}{\partial x}\right|_{y=\eta} - \sum_{j=1}^{L}\int_{y_{j-1}}^{y_j} \frac{\partial}{\partial y}\left[K_y(x,y)\frac{\partial \phi}{\partial y}\right]\mathrm{d}y$$
$$= K_x(y)\frac{\partial \eta}{\partial x}\left.\frac{\partial \phi}{\partial x}\right|_{y=\eta} + K_y(x,0)\left.\frac{\partial \phi}{\partial y}\right|_{y=0}$$
$$- K_y(x,\eta)\left.\frac{\partial \phi}{\partial y}\right|_{y=\eta}, \tag{7}$$

where $y_0 = 0$, $y_L = \eta(x,t)$, y_j $(j = 1, \dots, L-1)$ are the locations of the interfaces of different layers. Substituting the kinematic boundary condition (2g) and the no-flow boundary condition (2d) into (7), one has

$$\frac{\partial^2 W}{\partial x^2} = S_y \frac{\partial \eta}{\partial t}. \tag{8}$$

2.3. Definition of the mean water levels

Assume that the periods [T] of components of the sea tide equal rational numbers. Let $P_j = m_j/n_j$ $(j = 1, \dots, N)$

be the period of the jth sinusoidal component, where m_j and n_j are two positive integers prime to each other and P_j is measured in hours. Then, with unit of h^{-1}, the angular velocity of the jth sinusoidal component is

$$\omega_j = \frac{2\pi}{P_j} = \frac{2\pi n_j}{m_j}, \quad j = 1, \ldots, N. \tag{9a}$$

Therefore, the tide water level

$$W_{\text{Tide}}(t) = \sum_{j=1}^{N} A_j \cos(\omega_j t + c_j) \tag{9b}$$

is periodic with respect to the time t with a period of P (in h) given by

$$P = \frac{\min_{\text{CM}}(m_1, \ldots, m_N)}{\max_{\text{CD}}(n_1, \ldots, n_N)}, \tag{9c}$$

where $\min_{\text{CM}}(m_1, \ldots, m_N)$ denotes the minimum common multiple of $m_1, \ldots, m_N$, and $\max_{\text{CD}}(n_1, \ldots, n_N)$ the maximum common divisor of $n_1, \ldots, n_N$. In fact, because $n_j P / m_j$ is an integer for each $j = 1, \ldots, N$, it follows that

$$\begin{aligned} W_{\text{Tide}}(t+P) &= \sum_{j=1}^{N} A_j \cos(\omega_j t + \omega_j P + c_j) \\ &= \sum_{j=1}^{N} A_j \cos\left(\omega_j t + 2\pi \frac{n_j}{m_j} P + c_j\right) \\ &= \sum_{j=1}^{N} A_j \cos(\omega_j t + c_j) = W_{\text{Tide}}(t). \end{aligned} \tag{9d}$$

Based on (9d), one can assume that the solution of the steady-periodic nonlinear free surface problem (2a)–(2g) will also be a periodic function of time t with a period of P. This assumption is physically reasonable because the periodic function (2c) is the only driving force of the problem. Therefore, it is reasonable to define the mean water levels of problem (2a)–(2g) by

$$\bar{\phi}(x, y) = \frac{1}{P} \int_t^{t+P} \phi(x, y, \tau) \, d\tau, \tag{10a}$$

$$\bar{\eta}(x) = \frac{1}{P} \int_t^{t+P} \eta(x, \tau) \, d\tau, \tag{10b}$$

$$\begin{aligned} \overline{W}(x) &= \frac{1}{P} \int_t^{t+P} W(x, \tau) \, d\tau \\ &= \frac{1}{P} \int_t^{t+P} \left\{ \int_0^{\eta(x,\tau)} K_x(y) [\phi(x, y, \tau) - y] \, dy \right\} d\tau. \end{aligned} \tag{10c}$$

3. Asymptotic properties of the solution to the free surface problem

Due to the nonlinearity of the free surface problem, it is difficult to find the analytical solution. Nevertheless, some asymptotic properties of the solution when $x \to \infty$ can be found and will be discussed as follows.

Integrating (8) in the interval $(t, t+P)$ with respect to time t, dividing the resulting equation by P, and using the periodicity assumption of $\phi(x, y, t)$, $\eta(x, t)$ and $W(x, t)$, yield

$$\frac{d^2 \overline{W}}{dx^2} = 0, \quad 0 < x < \infty. \tag{11a}$$

From (4) and (6) one obtains

$$\overline{W}(0) = \frac{1}{P} \int_t^{t+P} \left\{ \int_0^{W_{\text{Tide}}(t)} K_x(y) [W_{\text{Tide}}(t) - y] \, dy \right\} dt, \tag{11b}$$

$$\lim_{x \to \infty} \frac{\partial \overline{W}}{\partial x} = 0. \tag{11c}$$

The right-hand side of (11b) is a constant determined by the horizontal hydraulic conductivity $K_x(y)$ and the tide water level $W_{\text{Tide}}(t)$. Therefore, the solution of (11a)–(11c) is equal to the following constant:

$$\overline{W}(x) \equiv \frac{1}{P} \int_t^{t+P} \left\{ \int_0^{W_{\text{Tide}}(t)} K_x(y) [W_{\text{Tide}}(t) - y] \, dy \right\} dt, \quad 0 \leqslant x < \infty. \tag{12}$$

When $x \to \infty$, the tide-induced oscillations will die out. Therefore, the piezometric head $\phi(\infty, y, t)$ and the free surface $y = \eta(\infty, t)$ will become independent of time t, namely,

$$\lim_{x \to \infty} \phi(x, y, t) = \phi_\infty(y), \quad \lim_{x \to \infty} \eta(x, t) = \eta_\infty, \tag{13a}$$

where η_∞ is an unknown constant. Moreover, because the tidal oscillation is the only driving force of the free surface problem, the piezometric head $\phi(x, y, t)$ distribution will tend to hydrostatic state as $x \to \infty$, this implies that

$$\phi_\infty(y) = \eta_\infty, \quad 0 < y \leqslant \eta_\infty. \tag{13b}$$

Combining Eqs. (10c), (13a) and (13b) yields

$$\begin{aligned} \lim_{x \to \infty} \overline{W}(x) &= \lim_{x \to \infty} \frac{1}{P} \int_t^{t+P} \left[\int_0^{\eta(x,\tau)} K_x(y) [\phi(x, y, \tau) - y] \, dy \right] d\tau \\ &= \frac{1}{P} \int_t^{t+P} \left[\int_0^{\eta_\infty} K_x(y) (\eta_\infty - y) \, dy \right] d\tau \\ &= \int_0^{\eta_\infty} K_x(y) (\eta_\infty - y) \, dy. \end{aligned} \tag{14}$$

From (12) and (14), one obtains

$$\int_0^{\eta_\infty} K_x(y) (\eta_\infty - y) \, dy = \frac{1}{P} \int_t^{t+P} \left\{ \int_0^{W_{\text{Tide}}(t)} K_x(y) [W_{\text{Tide}}(t) - y] \, dy \right\} dt. \tag{15a}$$

Using Eq. (15a), one can analyze the value range of the asymptotic watertable η_∞ and the relationship between η_∞ and the other model parameters. An important and obvious fact implied in Eq. (15a) is that the asymptotic watertable η_∞ is independent of the vertical conductivity $K_y(x,y)$ of the aquifer.

Using the notation $\overline{W}_{\mathrm{Tide}}(t) = W_{\mathrm{Tide}}(t) - b$, Eq. (15a) can be rewritten as

$$\begin{aligned}\int_0^{\eta_\infty} K_x(y)(\eta_\infty - y)\,\mathrm{d}y &= \frac{1}{P}\int_t^{t+P}\left\{\int_0^{b+\overline{W}_{\mathrm{Tide}}} K_x(y)[b+\overline{W}_{\mathrm{Tide}} - y]\,\mathrm{d}y\right\}\mathrm{d}t \\ &= \int_0^b K_x(y)(b-y)\,\mathrm{d}y \\ &\quad + \frac{1}{P}\int_t^{t+P}\left\{\int_b^{b+\overline{W}_{\mathrm{Tide}}} K_x(y)[b+\overline{W}_{\mathrm{Tide}} - y]\,\mathrm{d}y\right\}\mathrm{d}t. \end{aligned} \tag{15b}$$

Making a transform $\bar{y} = y - b$ to the integral variable in the last integral term of (15b), yields

$$\begin{aligned}\int_0^{\eta_\infty} K_x(y)(\eta_\infty - y)\,\mathrm{d}y &= \int_0^b K_x(y)(b-y)\,\mathrm{d}y \\ &\quad + \frac{1}{P}\int_t^{t+P}\left\{\int_0^{\overline{W}_{\mathrm{Tide}}} K_x(\bar{y}+b)[\overline{W}_{\mathrm{Tide}}(t) - \bar{y}]\,\mathrm{d}\bar{y}\right\}\mathrm{d}t. \end{aligned} \tag{15c}$$

Eq. (15c) implies that the asymptotic watertable η_∞ is greater than the mean sea level b for an arbitrary positive hydraulic conductivity $K_x(y)$. In fact, because $K_x(y) > 0$, the function $f(\lambda) \overset{\text{def.}}{=} \int_0^\lambda K_x(y)(\lambda - y)\,\mathrm{d}y$ increases with λ strictly. Moreover, the last integral term in (15c) is positive because the inside integral $\int_0^{\overline{W}_{\mathrm{Tide}}} K_x(\bar{y}+b)[\overline{W}_{\mathrm{Tide}}(t) - \bar{y}]\,\mathrm{d}\bar{y}$ is always positive for an arbitrarily fixed, nonzero value of $\overline{W}_{\mathrm{Tide}}(t)$, no matter it is positive or negative. Therefore, one has $\eta_\infty > b$. If $K_x(y)$ is given, one may evaluate the difference $\Delta = \eta_\infty - b$ based on Eq. (15a). In the next section several examples will be given.

4. Application

In this section, Eq. (15a) will be used to investigate the asymptotic watertable η_∞ for the commonplace aquifers with typical varying patterns of the horizontal hydraulic conductivity $K_x(y)$. The vertical hydraulic conductivity $K_y(x,y)$ can be arbitrary because η_∞ is independent of it. Comparison with existing results and discussion will be made also.

4.1. Type 1: aquifers with constant K_x

As the simplest case, assume that the horizontal hydraulic conductivity K_x of the aquifer is constant, then the left-hand side of (15a) is

$$\int_0^{\eta_\infty} K_x(\eta_\infty - y)\,\mathrm{d}y = \frac{1}{2}K_x\eta_\infty^2, \tag{16a}$$

evaluating the right-hand side of (15a) by means of the orthogonality of the different sinusoidal components in (2c) yields

$$\begin{aligned}\frac{1}{P}\int_t^{t+P}\left\{\int_0^{W_{\mathrm{Tide}}(t)} K_x[W_{\mathrm{Tide}}(t) - y]\,\mathrm{d}y\right\}\mathrm{d}t \\ = \frac{1}{2}K_x\left(b^2 + \frac{1}{2}\sum_{j=1}^N A_j^2\right). \end{aligned} \tag{16b}$$

From (15a), (16a) and (16b) one has

$$\eta_\infty = \sqrt{b^2 + \frac{1}{2}\sum_{j=1}^N A_j^2}. \tag{17}$$

Eq. (17) shows that the watertable at inland places far from the coastline will be higher than the mean sea level by a constant given by

$$\Delta_1 = \eta_\infty - b = \sqrt{b^2 + \frac{1}{2}\sum_{j=1}^N A_j^2} - b. \tag{18}$$

This is a generalization of the result (Eq. (1)) of Philip [7] in the sense that the free surface model here considers the vertical groundwater flow and is more general than the Boussinesq equation, on which Philip's [7] result is based. Moreover, Eq. (18) includes all the sinusoidal components of the sea tide but the results of Philip [7] only included single sinusoidal tidal component. Eq. (18) is valid for arbitrary vertical hydraulic conductivity $K_y(x,y)$ of the aquifer. Therefore, it also generalizes the result of Knight [2].

When $N = 1$, Eq. (17) is the same as the several previously published analytical solutions. For example, it is the same as the analytical solution of Barry et al. [1] after ignoring the capillary effects (Eq. (21) and (43) of their paper), it is the same as Eqs. (7) and (40) of Parlange et al. [6]. Using Taylor's expansion, Eq. (17) in the case of $N = 1$ can be approximated by $b + A_1^2/(4b)$, which is the same as the asymptotic limit of Eq. (24) of Nielsen [5] when $x \to \infty$.

The agreements between the experiment data available in the published literature and Eq. (17) are also satisfactory. For example, in the experiment of Smiles and Stokes [10], both the amplitude of the water level oscillation in the reservoir and the mean depth of the reservoir equal 7.5 cm (page 29 of their paper), i.e., $A_1 = b = 0.075$ m, $N = 1$. Therefore, according to (17), the normalized asymptotic elevation with respect to the tidal amplitude A_1 is then given by

$$Z^* = \eta_\infty / A_1 = \frac{b}{A_1}\sqrt{1 + \frac{1}{2}\frac{A_1^2}{b^2}} = \sqrt{1 + 0.5} \approx 1.23.$$

This is in line with the experiment results shown in Fig. 4 of Smiles and Stokes [10].

For the experiment of Parlange et al. [6], the amplitude of the water level oscillation in the reservoir is 9.0 cm, the mean depth of the reservoir equal 27.6 cm (page 261–262 of their paper), i.e., $A_1 = 9$ cm, $b = 27.6$ cm, $N = 1$. Therefore, according to (17), the asymptotic elevation for great x is then given by

$$\eta_\infty = \sqrt{b^2 + A_1^2/2} = 28.32 \text{ cm}.$$

This prediction is very close to the experiment result of 28.4 cm for the greatest value of x in Fig. 4 of their paper.

4.2. Type 2: multi-layered aquifer with piecewise constant K_x

Assume that the aquifer is composed of three different layers. The respective horizontal hydraulic conductivities of the lower, middle and upper layers are three constants K_{x1}, K_{x2} and K_{x3}. The thicknesses of the lower and middle layers are b_1 and b_2, respectively. The depth of the upper layer below the mean sea level, b_3, is great enough so that the tidal water remains connected with the upper layer at low tide (see Fig. 2). This assumption guarantees that any possible free surfaces are always in the upper layer so that the specific yield S_y in the kinematic free surface boundary condition (2g) to be constant. Then, the left-hand side of (15a) is

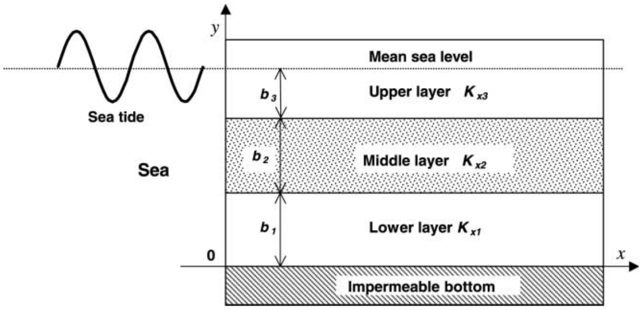

Fig. 2. Schematic of a multi-layered aquifer system near open tidal water.

$$\begin{aligned}
&\int_0^{\eta_\infty} K_x(y)(\eta_\infty - y)\,\mathrm{d}y \\
&= K_{x1}\int_0^{b_1}(\eta_\infty - y)\,\mathrm{d}y + K_{x2}\int_{b_1}^{b_1+b_2}(\eta_\infty - y)\,\mathrm{d}y \\
&\quad + K_{x3}\int_{b_1+b_2}^{\eta_\infty}(\eta_\infty - y)\,\mathrm{d}y = \frac{1}{2}K_{x3}\eta_\infty^2 + [(K_{x1} - K_{x3})b_1 \\
&\quad + (K_{x2} - K_{x3})b_2]\eta_\infty + c,
\end{aligned} \tag{19a}$$

where

$$c = \frac{1}{2}[(K_{x3} - K_{x2})(b_1 + b_2)^2 + (K_{x2} - K_{x1})b_1^2]. \tag{19b}$$

The right-hand side of (15a) can be evaluated by means of (2c) and (17) and equals

$$\begin{aligned}
&\frac{1}{P}\int_t^{t+P}\left[\int_0^{W_{\mathrm{Tide}}(t)} K_x(y)[W_{\mathrm{Tide}}(t) - y]\,\mathrm{d}y\right]\mathrm{d}t \\
&= \frac{K_{x3}}{4}\left(2b^2 + \sum_{j=1}^{N} A_j^2\right) + [K_{x1}b_1 + K_{x2}b_2 \\
&\quad - K_{x3}(b_1 + b_2)]b + c.
\end{aligned} \tag{19c}$$

Using (15a), (19a), (19c), and the equality $b = b_1 + b_2 + b_3$, one obtains

$$\begin{aligned}
&\frac{1}{2}K_{x3}(\eta_\infty - b)^2 + (K_{x1}b_1 + K_{x2}b_2 + K_{x3}b_2)(\eta_\infty - b) \\
&\quad - \frac{1}{4}K_{x3}\sum_{j=1}^{N} A_j^2 = 0.
\end{aligned} \tag{19d}$$

At the end of Section 3 it has been shown that $\eta_\infty - b > 0$, therefore, the negative root of (19d) should be neglected and the positive root is given by

$$\begin{aligned}
\Delta_2 &= \eta_\infty - b \\
&= \sqrt{\left(\sum_{j=1}^{3}\frac{K_{xj}b_j}{K_{x3}}\right)^2 + \frac{1}{2}\sum_{j=1}^{N} A_j^2} - \sum_{j=1}^{3}\frac{K_{xj}b_j}{K_{x3}}.
\end{aligned} \tag{20}$$

If the middle layer is semipermeable so that the horizontal transmissivity $K_{x2}b_2$ of the middle layer is negligible compared with the transmissivities $K_{x1}b_1$ and $K_{x3}b_3$ of the lower and upper layers, Eq. (20) can be simplified into

$$\Delta_2 = (b_3 + K_{x1}b_1/K_{x3})(\sqrt{1 + \alpha_L/2} - 1) \tag{21a}$$

where α_L is a dimensionless parameter given by

$$\alpha_L = \frac{K_{x3}^2}{(K_{x1}b_1 + K_{x3}b_3)^2}\sum_{i=1}^{N} A_i^2. \tag{21b}$$

If the aquifer comprises L different layers, and the depth of the top layer below the mean sea level is great enough so that the tidal water remains connected with the top layer at low tide, then Eq. (20) can be easily generalized into

$$\Delta_3 = \eta_\infty - b = \sqrt{\left(\sum_{j=1}^{L}\frac{K_{xj}b_j}{K_{xL}}\right)^2 + \frac{1}{2}\sum_{j=1}^{N}A_j^2} - \sum_{j=1}^{L}\frac{K_{xj}b_j}{K_{xL}}, \tag{22}$$

where K_{xL} is the horizontal hydraulic conductivity of the top layer and b_L is the depth of the top layer below the mean sea level, K_{xj}, b_j ($j = 1, \ldots, L-1$) are the horizontal hydraulic conductivities and thicknesses of the other layers.

4.3. Type 3: $K_x(y)$ decreasing with the depth

Aquifers of decreasing horizontal hydraulic conductivity with the depth are commonplace in reality [10]. For the sake of simplicity, assume that the horizontal hydraulic conductivity $K_x(y)$ is linearly decreasing with the depth and becomes zero at $y = 0$, i.e., $K_x(y) = ky$ with k being a positive constant. In this case, strictly speaking, the specific yield S_y of the aquifer will change with the depth. In order to use Eq. (15a), which is based on the assumption that S_y is constant on all the possible free surfaces corresponding to any tidal level, assume that the variation of S_y in the aquifer's elevation range $E_H \geqslant y \geqslant E_L$ is negligible, where E_H and E_L are the elevations of the tidal levels at high and low tides, respectively. Based on this assumption, (15a) can be used to estimate the asymptotic watertable. The left-hand side of (15a) is

$$\int_0^{\eta_\infty} K_x(y)(\eta_\infty - y)\,dy = k\int_0^{\eta_\infty} y(\eta_\infty - y)\,dy = \frac{1}{6}k\eta_\infty^3. \tag{23a}$$

Transforming the products of the trigonometric functions into sums and differences of the trigonometric functions, and using the orthogonality of different terms, the right-hand side of (15a) can be evaluated by means of (2c) and equals

$$\frac{1}{P}\int_t^{t+P}\left\{\int_0^{W_{\text{Tide}}(t)} ky[W_{\text{Tide}}(t) - y]\,dy\right\}dt = \frac{kb}{6}\left(b^2 + \frac{3}{2}\sum_{j=1}^{N}A_j^2\right). \tag{23b}$$

Using (15a), (23a) and (23b), it follows that

$$\eta_\infty = \left(b^3 + \frac{3b}{2}\sum_{j=1}^{N}A_j^2\right)^{1/3}. \tag{23c}$$

Therefore, the difference between the asymptotic watertable and the mean sea level is given by

$$\Delta_4 = \eta_\infty - b = \left(b^3 + \frac{3b}{2}\sum_{j=1}^{N}A_j^2\right)^{1/3} - b. \tag{24}$$

If there are two aquifers, one is of the type 1 and the other is of type 3. The tide amplitudes and the depth b of the two aquifers below the mean sea level are the same. Then Eqs. (18) and (24) imply that the difference between the asymptotic watertable and the mean sea level of the type 3 will be much greater than that of the type 1 aquifer. The former, Δ_4, is approximately twice as much as the later, Δ_1. In fact, using Taylor expansion

$$(1+\alpha)^p = 1 + p\alpha - \frac{p(1-p)\alpha^2}{2(1+\xi)^{2-p}}, \quad 0 < \xi < \alpha, \tag{25a}$$

one has

$$\Delta_1 = b\left(\sqrt{1 + \frac{1}{2b^2}\sum_{j=1}^{N}A_j^2} - 1\right) \approx \frac{1}{4b}\sum_{j=1}^{N}A_j^2, \tag{25b}$$

$$\Delta_4 = b\left[\left(1 + \frac{3}{2b^2}\sum_{j=1}^{N}A_j^2\right)^{1/3} - 1\right] \approx \frac{1}{2b}\sum_{j=1}^{N}A_j^2 \approx 2\Delta_1. \tag{25c}$$

For example, when $\sum_{j=1}^{N}A_j^2 = 1$ m, $b = 3$ m, from (24) and (18) one obtains $\Delta_4 = 0.158$ m, $\Delta_1 = 0.082$ m. As explained by Knight [2], it is the watertable-dependent transmissivity of the unconfined aquifer that causes a higher mean watertable than the mean sea level. The effective transmissivity of the aquifer when water flows into the aquifer at high tide is greater than that when water flows out from the aquifer at low tide. For steady periodic state when there is neither seawater intrusion nor net inland recharge, to maintain water balance, water entering a coastal unconfined aquifer at high tide should be exactly equal to that leaving the aquifer at low tide. Consequently, only when the mean watertable stays above the mean sea level, can the unconfined aquifer maintain correct water balance (see Fig. 1). As the aquifer of type 3 is concerned, the horizontal hydraulic conductivity decreasing with the depth enhances the existing transmissivity difference at high and low tides, and consequently it enhances the asymptotic watertable.

5. Conclusions

The influence of the tidal water level fluctuation on the mean watertable in an unconfined coastal aquifer is investigated. The aquifer is assumed to be inhomogeneous and anisotropic with the two principle directions of the hydraulic conductivities being horizontal and vertical. The study is based on a nonlinear free surface model under five assumptions: the horizontal hydraulic conductivity depends on the aquifer depth only; the water–land boundary is vertical; there is no seepage face; there is no net inland recharge; and the specific storage is constant wherever the free surfaces are located. Due to nonlinearity of the free-surface problem, the superposition principle is not applicable. Hence a multi-sinusoidal-component tidal boundary condition is considered at the coastline. An integral equation satisfied by the asymptotic watertable as the landward distance approaches infinity is derived. This equation leads to the following findings. (1) The asymptotic watertable is determined by the horizontal hydraulic conductivity, the depth of the aquifer below the mean sea level, and the tidal parameters. It is independent of the vertical hydraulic conductivity of the aquifer. (2) The asymptotic watertable is greater than the mean sea level even in the absence of net inland recharge. For given horizontal hydraulic conductivity the asymptotic watertable can be solved explicitly from the integral equation. Commonplace aquifers are considered such as multi-layered aquifers, and aquifers whose horizontal hydraulic conductivity decreases linearly with depth. For these aquifers the asymptotic watertables are also independent of the frequencies and phase shifts of the sinusoidal components of the tide. It is also found that the difference between the asymptotic watertable and mean sea level will be significantly enhanced when the horizontal hydraulic conductivity decreases with depth. Comparison with previous studies found that the results of Philip [7], Knight [2], Parlange et al. [6], and Barry et al. [1], are special cases of this paper. The experiment data of Smiles and Stokes [10] and Parlange et al. [6] are explained with the analytical predictions of this paper.

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