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Analytical solutions of tidal groundwater flow in coastal two-aquifer system

Hailong Li [a,b,*], Jiu Jimmy Jiao [a]

[a] Department of Earth Sciences, The University of Hong Kong, Hong Kong, PR China

[b] Institute of Biomathematics, Anshan Normal College, Anshan 114005, Liaoning, PR China

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Abstract

This paper presents a complete analytical solution to describe tidal groundwater level fluctuations in a coastal subsurface system. The system consists of two aquifers and a leaky layer between them. Previous solutions of Jacob [Flow of groundwater, in: H. Rouse (Ed.), Engineering Hydraulics, Wiley, New York, 1950, pp. 321–386], Jiao and Tang [Water Resour. Res. 35 (3) (1999) 747], Li and Jiao [Adv. Water Resour. 24 (5) (2001a) 565], Li et al. [Water Resour. Res. 37 (2001) 1095] and Jeng et al. [Adv. Water Resour. (in press)] are special cases of the new solution. The present solution differs from previous work in that both the effects of the leaky layer's elastic storage and the tidal wave interference between the two aquifers are considered. If the upper and lower aquifers have the same storativities and transimissivities, the system can be simplified into an equivalent double-layered, aquifer–aquitard system bounded by impermeable layers from up and down. It is found that the leaky layer's elastic storage behaves as a buffer to the tidal wave interference between the two aquifers. The buffer capacity increases with the leaky layer's thickness, specific storage, and decreases with the leaky layer's vertical permeability. Great buffer capacity can result in negligible tidal wave interference between the upper and lower aquifers so that the Li and Jiao (loc. cit.) solution applies.

1. Introduction

With the social and economic development in coastal areas, various coastal hydrogeological, engineering, and environmental problems arise. Among them are, for example, seawater intrusion, stability of coastal engineering structures, beach dewatering for construction purposes, and deterioration of the marine environment. In order to solve these problems, it is necessary to identify the coastal hydrogeological conditions, which include aquifer parameter estimation and interaction between groundwater and seawater. In coastal aquifers, the groundwater level (hydraulic head or water table) fluctuates with time in response to the water level fluctuations of the tidal water body (ocean or river). This natural phenomenon has the potential to provide a convenient, economic and reliable way of identifying coastal hydrogeological conditions in large scale. This is maybe one of the main reasons that the study on the dynamic relation between seawater and coastal groundwater has become an active research area since the 1950s. For example, Jacob [5], Nielsen [14], Li and Chen [9], Sun [17] derived various solutions to describe the tidal groundwater fluctuations in a single coastal aquifer. In many coastal areas, there are more than two aquifers separated by semipermeable layer(s) (e.g., [2,7,10,11,15,16,18]). Van der Kamp [18] studied a three-layered coastal aquifer system consisting of one aquifer bounded by two semipermeable layers from up and down. Jiao and Tang [7] considered the groundwater head fluctuations in the confined aquifer of a coastal multi-layered groundwater system consisting of a confined aquifer, an unconfined aquifer, and a leaky layer between them. They ignored the leaky layer's elastic storage and tide-induced water table variations in the unconfined aquifer. Li and Jiao [10] improved the result of Jiao and Tang [7] by taking the leaky layer's elastic storage into account. Li et al. [13] used a perturbation method to investigate the tidal wave interference between the unconfined and confined aquifers, but they ignored the effects of the elastic storage of the leaky layer. Jeng et al. [6] presented a complete analytical solution to describe the tidal wave propagation and

* Corresponding author. Tel.: +86-412-5841417; fax: +852-25176912.

E-mail addresses: hlica@hkusua.hku.hk (H. Li), jjiao@hku.hk (J.J. Jiao).

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interference in the unconfined and confined aquifers separated by a thin, no-storativity leaky layer.

According to various pumping test data available in the literature on leaky confined aquifer systems, if the leaky layer is composed of thick, soft sedimentary materials, its elastic storage is much greater than that of the aquifers [10]. In this case, the influence of the leaky layer's elastic storage on the tidal wave propagation behavior in the coastal aquifers has to be considered. It is also worthwhile to investigate the tidal wave interference between different aquifers. Based on this motivation, this paper presents an analytical solution that describes the tidal wave propagation in the two aquifers separated by a leaky layer with significant elastic storage. After the analytical solution is derived, the tidal wave interference between the two aquifers via the leaky layer is analyzed. Special cases in which the previous simple solutions apply are discussed.

2. Mathematical model and analytical solution

2.1. Mathematical model

Consider a subsurface system consisting of two aquifers and a leaky layer between them. Assume that (a) each layer is homogeneous and horizontal, (b) vertical flows in the aquifers are negligible, and (c) all formations have a clear-cut vertical water-land boundary. These assumptions have been used in many previous studies (e.g., [5,7,10,17]. Assumption (b) was proposed by Hantush [4] for well-ward groundwater flow and again used by Neuman and Witherspoon [1969].

Choose an x–z coordinate system so that the x-axis is landward positive, and the z-axis be vertical, upward positive. The origin is the crossover point of the leaky layer's bottom line and the vertical water-land boundary (see Fig. 1). The lower aquifer has an impermeable bottom. The upper aquifer may be either a confined aquifer with an impermeable roof or an unconfined aquifer with its depth below the mean sea level being

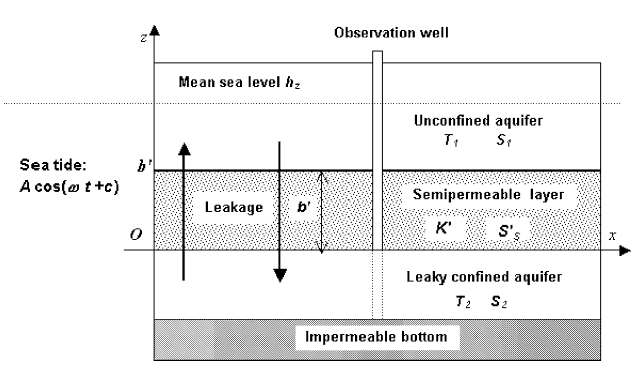

Fig. 1. A coastal two-aquifer system when the upper aquifer is unconfined.

much greater than the tidal amplitude. The groundwater flow equations in the two aquifers are [4,10]

$$S_j\frac{\partial H_j}{\partial t} = T_j\frac{\partial^2 H_j}{\partial x^2} \pm K'\frac{\partial h}{\partial z}\bigg|_{z=Z_j}, \qquad -\infty < t < \infty,\ x > 0,\ j = 2, 1, \tag{1a}$$

where S_j, T_j and H_j $(j = 1, 2)$ are the storativity (dimensionless), transmissivity $[\mathrm{L^2T^{-1}}]$ and hydraulic heads [L] of the two aquifers, respectively; the subscript $j = 1$ corresponds to the upper aquifer and $j = 2$ to the lower one; $h(x, z, t), K'$ are the groundwater head [L], vertical permeability $[\mathrm{LT^{-1}}]$ of the leaky layer, respectively; $Z_1 = b'$, $Z_2 = 0$, b' is the leaky layer's thickness [L]. In case that the upper aquifer is unconfined, the nonlinear Boussinesq equation for unconfined aquifer is linearized based on the assumption of small ratio of tidal amplitude to the aquifer's depth below the mean sea level [1]. On the ocean-land boundary $x = 0$, the tidal boundary conditions

$$H_j(0, t) = A\cos(\omega t) = A\,\mathrm{Re}[\exp(\mathrm{i}\omega t)], \quad j = 1, 2 \tag{1b}$$

are used, where $\mathrm{i} = \sqrt{-1}$, Re denotes the real part of the followed complex expression, A and ω are the tidal amplitude [L] and frequency $[\mathrm{T^{-1}}]$, respectively. The datum of the hydraulic heads of the aquifers is set to be the mean sea level. In inland places far from the coastline, no-flow boundary condition is used, i.e.

$$\lim_{x\to\infty}\frac{\partial H_j}{\partial x} = 0, \quad j = 1, 2. \tag{1c}$$

The groundwater flow in the leaky layer is two-dimensional and the groundwater head $h(x, z, t)$ should satisfy the following differential equation and boundary conditions:

$$S'_S\frac{\partial h}{\partial t} = K'\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2}\right), \qquad -\infty < t < \infty,\ 0 < z < b',\ 0 < x < \infty, \tag{2a}$$

$$h(x, b', t) = H_1(x, t), \tag{2b}$$

$$h(x, 0, t) = H_2(x, t), \tag{2c}$$

where S'_S is the specific storativity $[\mathrm{L^{-1}}]$ of the leaky layer. Direct consideration of (2a) will make the problem too complicated to derive analytical solutions. As a simplification similar to Hantush [4], neglecting the term $(\partial^2 h/\partial x^2)$ in (2a) leads to

$$S'_S\frac{\partial h}{\partial t} = K'\frac{\partial^2 h}{\partial z^2}, \quad -\infty < t < \infty,\ 0 < z < b'. \tag{2d}$$

In Appendix D it is shown that the solution of (2d) is an approximate solution to (2a) in the sense that $(\partial^2 h/\partial x^2)$ is negligible compared with $(\partial^2 h/\partial z^2)$. The following quantitative assumptions are used in Appendix D:

$$\frac{T_j S'_S}{K' S_j} = \frac{T_j S'}{T' S_j} \gg 1 \quad (j = 1, 2), \tag{2e}$$

$$\frac{\omega S'_S b' T_j}{K'^2} = \frac{T_j}{u' T'} \gg 1 \quad (j = 1, 2), \tag{2f}$$

where T' and S' are the transmissivity and storativity of the semipermeable layer, $u' = (K'/(\omega S'_S b'^2)) = (L/(\omega S'))$. Li and Jiao [10] reviewed pumping test data available in the literature and concluded that for most leaky aquifer systems one has $T_j \gg T'$, $S' \gg S_j$ $(j = 1, 2)$, so assumption (2e) is reasonable. According to Table 1 of Li and Jiao [10], the range of $u = L/(\omega S_j)$ is 0–10.0, so $u' = uS/S' \leqslant \mathrm{O}(1)$ due to $S' \gg S_j$ $(j = 1, 2)$. This justifies assumption (2f). The validity of the assumption (b) and Eq. (2d) for the case of coastal aquifer systems is also examined numerically by Li and Jiao [12].

2.2. Analytical solutions

Only the expression of the groundwater heads $H_j(x, t)$ $(j = 1, 2)$ in the two aquifers will be presented here because for practical groundwater flow problems they are the major concerns compared to the groundwater head $h(x, z, t)$ in the leaky layer. Details about $h(x, z, t)$ and the solution derivation are presented in Appendix A.

2.2.1. Definition of basic parameters

There are seven aquifer parameters involved in model (1a)–(1c), (2b)–(2d), namely, $S_1, T_1, S'_S, K', b', S_2$ and T_2 (see symbols in Fig. 1). They can be grouped into five basic, independent parameters that can determine the solutions. The first is the so-called buffer capacity (dimensionless) of the leaky layer defined as

$$\theta = b' \sqrt{\frac{\omega S'_S}{2K'}}. \tag{3a}$$

This parameter describes the capacity of the leaky layer to buffer the tidal wave interference between the two aquifers. The greater the buffer capacity θ is, the smaller the interference will be. The next two parameters are the dimensionless leakage–storativity ratios of the two aquifers given by

$$u_j = \frac{L}{\omega S_j}, \quad j = 1, 2, \tag{3b}$$

where $L = K'/b'$ $[\mathrm{T}^{-1}]$ is the leakance of the leaky layer [4]. The last two parameters are the two aquifer's tidal propagation parameters $[\mathrm{L}^{-1}]$ defined as

$$a_j = \sqrt{\frac{\omega S_j}{2T_j}}, \quad j = 1, 2. \tag{3c}$$

If the middle layer becomes completely impermeable $(K' = 0)$, the tidal wave propagation behaviors in the two aquifers are fully determined by a_j $(j = 1, 2)$ [5], i.e.,

$$H_j(x, t)\big|_{K'=0} = A \exp(-a_j x) \cos(\omega t - a_j x), \quad j = 1, 2. \tag{4}$$

In order to express the solution to (1a)–(1c), (2b)–(2d) succinctly, a pair of derived complex parameters is used. They are

$$B_j^2 = 2a_j^2 \{\mathrm{i} + u_j (1 + \mathrm{i}) \theta \coth[(1 + \mathrm{i})\theta]\}, \quad \mathrm{Re} B_j > 0, \quad j = 1, 2. \tag{5}$$

The proof of $\mathrm{Re} B_j > 0$ $(j = 1, 2)$ is given in Appendix B. The parameter B_1 is dependent on the upper aquifer and leaky layer and independent of the lower aquifer. It characterizes the tidal wave propagation in the upper aquifer if the influence of tidal wave propagation in the lower aquifer is neglected. Symmetrically, the parameter B_2 is dependent on the lower aquifer and the leaky layer and independent of the upper aquifer. It characterizes the tidal wave propagation in the lower aquifer if the influence of tidal wave propagation in the upper aquifer is neglected.

2.2.2. Solutions when $\lambda_1 \neq \lambda_2$ and $B_1 \neq B_2$

With above parameters, the solutions of (1a)–(1c), (2b)–(2d) can be written as

$$H_j(x, t) = \frac{A}{2} \mathrm{Re}\left\{ \left[(1 + \alpha_j) \mathrm{e}^{-\lambda_j x} + (1 - \alpha_j) \mathrm{e}^{-\lambda_{3-j} x} \right] \mathrm{e}^{\mathrm{i}\omega t} \right\}, \quad j = 1, 2 \text{ if } \lambda_1 \neq \lambda_2 \text{ and } B_1 \neq B_2, \tag{6a}$$

where

$$\lambda_{1,2}^2 = \frac{1}{2} \left[B_1^2 + B_2^2 \pm \mathrm{csgn}(z_c) \sqrt{z_c} \right], \quad \mathrm{Re}\, \lambda_{1,2} > 0, \tag{6b}$$

$$\alpha_j = \frac{B_j^2 - B_{3-j}^2 - 2\varepsilon_j}{\lambda_j^2 - \lambda_{3-j}^2}, \quad j = 1, 2, \tag{6c}$$

in which

$$\varepsilon_j = \frac{2a_j^2 u_j (1 + \mathrm{i})\theta}{\sinh[(1 + \mathrm{i})\theta]}, \quad j = 1, 2, \tag{7a}$$

$$z_c = (B_1^2 - B_2^2)^2 + 4\varepsilon_1 \varepsilon_2. \tag{7b}$$

The function $\mathrm{csgn}(z_c)$ equals 1 if the complex number z_c lies either in the right half part of the complex plane or on the positive part of the imaginary axis plus the origin, otherwise, $\mathrm{csgn}(z_c) = -1$. It is not obvious that $\mathrm{Re}\, \lambda_{1,2} > 0$. In Appendix C it is shown that $\mathrm{Re}\, \lambda_{1,2} \neq 0$. Therefore, the root of the RHS of (6b) with positive real part can be chosen so that $\mathrm{Re}\, \lambda_{1,2} > 0$. The parameter ε_j $(j = 1, 2)$ is called the interference coefficient $[\mathrm{L}^{-2}]$ of the jth aquifer. The greater the module of ε_j is, the more easily the tidal wave propagation in the jth aquifer will be interfered by that in the other aquifer. Their product $\varepsilon_1 \varepsilon_2$ is called the interference capacity of the two-aquifer system. The greater the module of $\varepsilon_1 \varepsilon_2$ is, the more significantly the tidal wave propagation in the two aquifers

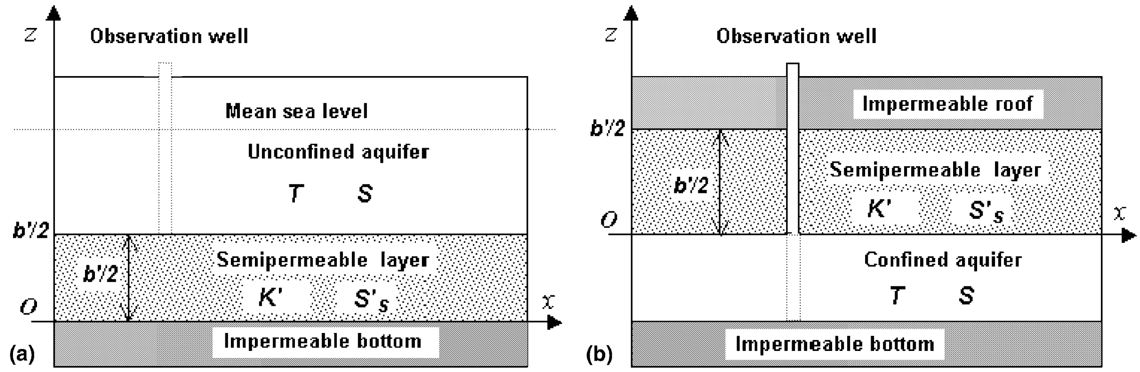

Fig. 2. Different configuration of coastal double-layered, aquifer–aquitard systems: (a) an unconfined aquifer overlying an aquitard and (b) a confined aquifer underlying an aquitard.

will be interfered by each other. Solution (6a) is well defined for situations when $\lambda_1 \neq \lambda_2$ and $B_1 \neq B_2$ because the denominator of (6c) does not equal zero.

2.2.3. Solution when $B_1 = B_2 = B$

Based on (5), (A.21) and (A.22), it follows that $B_1 = B_2 = B$ if and only if

$$\Theta_{\mathrm{re}}(\theta)(a_1^2u_1 - a_2^2u_2) + \mathrm{i}[a_1^2 - a_2^2 + \Theta_{\mathrm{im}}(\theta)(a_1^2u_1 - a_2^2u_2)] = 0. \tag{8}$$

Using the facts $\Theta_{\mathrm{im}}(\theta) > 0$, $\Theta_{\mathrm{re}}(\theta) > 1 > 0$ for any $\theta > 0$, and $\lim_{\theta\to 0} \Theta_{\mathrm{re}}(\theta) = 1$, $\lim_{\theta\to 0} \Theta_{\mathrm{im}}(\theta) = 0$, Eq. (8) implies that both $a_1^2u_1 = a_2^2u_2$ and $a_1^2 = a_2^2$ hold, which leads to $T_1 = T_2 = T$, $S_1 = S_2 = S$, and $\varepsilon_1 = \varepsilon_2 = \varepsilon$. Let $B_1, B_2 \to B$, $\varepsilon_1, \varepsilon_2 \to \varepsilon$ in (6a)–(6c), one obtains the solution when the upper and lower aquifers have the same storativities and transimissivities

$$H_1(x,t) = H_2(x,t) = A\,\mathrm{Re}[\exp(-x\sqrt{B^2-\varepsilon} + \mathrm{i}\omega t)] \quad \text{if } B_1 = B_2 = B. \tag{9}$$

Because Appendix C does not use the restriction $B_1 \neq B_2$, hence it also implies that $\mathrm{Re}\sqrt{B^2-\varepsilon} > 0$. An interesting fact is that solution (9) can be used to describe tidal groundwater flow in a double-layered aquifer–aquitard system. Using (9) and the symmetric property of the two-aquifer system, one can easily find that on the horizontal middle plane of the leaky layer $z = b'/2$, the vertical hydraulic gradient equals zero, i.e.,

$$\left.\frac{\partial h}{\partial z}\right|_{z=b'/2} = 0. \tag{10}$$

Eq. (10) can also be obtained by direct calculation using the expression of $h(x,z,t)$.

Eqs. (9) and (10) imply that the tidal wave propagation in the two aquifers behaves as if the horizontal middle plane of the leaky layer were impermeable. From this viewpoint, Eq. (9) describes the tidal wave propagation in a double-layered, aquifer–aquitard system bounded by impermeable layers on the top and at the bottom. The aquifer can either overlie or underlie the aquitard, which has a thickness of $b'/2$ (see Fig. 2).

2.2.4. Solution when $\lambda_1 = \lambda_2 = \lambda$

If $\lambda_1 = \lambda_2 = \lambda$, then from (6b) it follows that

$$(B_1^2 - B_2^2)^2 + 4\varepsilon_1\varepsilon_2 = 0, \tag{11a}$$

$$\lambda = \lambda_1 = \lambda_2 = \sqrt{(B_1^2 + B_2^2)/2}. \tag{11b}$$

In order to obtain the solutions when $\lambda_1 = \lambda_2 = \lambda$, let $\lambda_1, \lambda_2 \to \lambda$ in (6a), and use the limit equality

$$\lim_{\lambda_1,\lambda_2\to\lambda} \frac{\exp(-\lambda_1 x) - \exp(-\lambda_2 x)}{\lambda_1 - \lambda_2} = -x\exp(-\lambda x), \tag{12}$$

one obtains the solutions

$$H_j(x,t) = A\,\mathrm{Re}\left[\left(1 - \frac{B_j^2 - B_{3-j}^2 - 2\varepsilon_j}{4\lambda}x\right)\exp(-\lambda x + \mathrm{i}\omega t)\right], \quad j = 1,2 \text{ if } \lambda_1 = \lambda_2 = \lambda. \tag{13}$$

Because $\mathrm{Re}\,\lambda = \mathrm{Re}\,\lambda_1 = \mathrm{Re}\,\lambda_2 > 0$, the term $x\exp(-\lambda x)$ in (12) is bounded for $x > 0$ and tends to zero as x approaches infinite, hence solutions (13) satisfy the boundary conditions (1c). Eq. (11a) is the criterion to judge the case of $\lambda_1 = \lambda_2 = \lambda$. There do exist aquifer parameters that satisfy the judge criterion (11a) so that the tidal wave propagation in the upper and lower aquifers is described by the solution (13). For example, if the aquifer parameters satisfy $T_1 = T_2$, $S_1 = 81S_2$, $S'_S = 0$, $u_2 = 40$, then one can easily check that (11a) holds. Hence such an aquifer system can be described by solution (13).

It seems that the tidal wave propagation in the upper and lower aquifers will be identical if the upper and lower aquifers have the same eigenvalue $\lambda_1 = \lambda_2 = \lambda$. But the two different complex coefficients $1 - (B_j^2 - B_{3-j}^2 - 2\varepsilon_j)x/(4\lambda)$ $(j = 1,2)$ in (13) show that both the amplitudes and the phase shift of the solutions in the upper and lower aquifers are different. Therefore, the tidal waves in the upper and lower aquifers described by

(13) are not identical. Solution (13) is only a different form of the general solution (6a) when the removable singularity $\lambda_1 = \lambda_2 = \lambda$ occurs. To explain this more clearly, the general solution (6a), the solution (13) and Jacob's solution (4) are compared and discussed below. The following aquifer parameter values are used: $T_1 = T_2 = 1200\ \mathrm{m}^2\ \mathrm{d}^{-1}$, $S_1 = 81S_2 = 0.00405$, $\omega = 2\pi/12\ \mathrm{h}^{-1}$, $A = 1$ m. Fig. 3 shows how the amplitude changes with the landward distance for the general solution (6a) when $\theta = 0.5, u_2 = 40$, for the solution (13) when $\theta = 0, u_2 = 40$, and for Jacob's solution (4) when $u_1 = u_2 = 0$. When the middle layer is semipermeable and $\theta > 0, u_2 = 40$, the aquifer parameters do not satisfy the judge criterion (11a), so the groundwater flow is described by the general solution (6a). When the middle layer is semipermeable and $\theta = 0, u_2 = 40$, the aquifer parameters satisfy the judge criterion (11a), so the groundwater flow is described by the solution (13). When the middle layer is completely impermeable, i.e., $u_1 = u_2 = 0$, the groundwater flow is described by Jacob's solution (4). One can see that the amplitudes of the solutions in the upper and lower aquifers have significant discrepancy. But the amplitudes of the solution (13) and the general solution (6a) in each aquifer are very close to each other. In particular, the amplitude of the general solution in each aquifer when $\theta = 0.1$ is almost coincident with that of the solution (13) and it is not shown in Fig. 3.

Compared the solution (13) with Jacob's solution, the solution (13) has an average effect on the amplitudes in the upper and lower aquifers. For Jacob's solution, the tide-influenced distance of the lower aquifer is much longer than that of the upper aquifer because the diffusivity of the lower aquifer is much greater than that of the upper aquifer. For the solution (13), due to the leakance of the semipermeable layer, the tide-influenced distance of the upper aquifer is increased, and the tide-influenced distance of the lower aquifer is shortened.

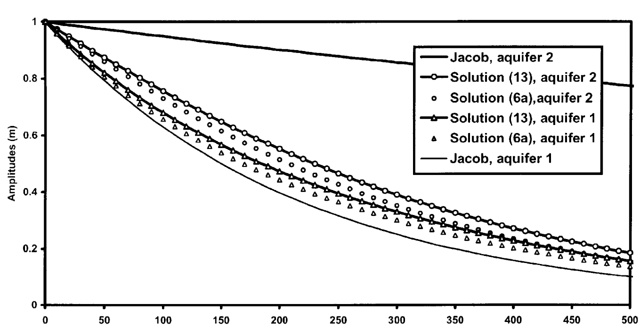

Fig. 3. Amplitude versus landward distance curves of general solution (6a) for $\theta = 0.5, u_2 = 40$, of solution (13) for $\theta = 0, u_2 = 40$, and of Jacob's solution (4) for $u_1 = u_2 = 0$ when $T_1 = T_2 = 1200\ \mathrm{m\ d}^{-1}$, $S_1 = 81S_2 = 0.00405$, $A = 1$ m, $\omega = 2\pi/12\ \mathrm{h}^{-1}$.

3. Approximate solutions under weak interference conditions

In this section, approximate solutions for small $|\varepsilon_j|$ $(j = 1, 2)$ will be given and discussed. Assume $B_1^2 - B_2^2 \neq 0$ and $|\varepsilon_j| \ll |B_1^2 - B_2^2|$ so that $\varepsilon_1\varepsilon_2$ is negligible compared with $B_1^2 - B_2^2$. Let $\varepsilon_1\varepsilon_2 = 0$ in (6b), one has

$$\lambda_j\big|_{\varepsilon_1\varepsilon_2=0} = B_j, \quad j = 1, 2. \tag{14}$$

Substituting (14) into (6a), yields

$$H_j(x,t) = A\,\mathrm{Re}\left\{ \mathrm{e}^{\mathrm{i}\omega t}\left[\mathrm{e}^{-B_j x} - \frac{\varepsilon_j(\mathrm{e}^{-B_j x} - \mathrm{e}^{-B_{3-j} x})}{B_j^2 - B_{3-j}^2} \right] \right\}, \quad j = 1, 2 \text{ if } |\varepsilon_1\varepsilon_2| \ll |B_1^2 - B_2^2|. \tag{15}$$

3.1. Comparison with existing solutions when $S_S' = 0$

Li et al. [13] and Jeng et al. [6] discussed the situation when the leaky layer's specific storage $S_S' = 0$, which is equivalent to $\theta = 0$ according to (3a). Let $\theta \to +0$ in (5) and (7a), and use (3b), (3c), one has

$$\lim_{\theta\to 0} B_j^2 = 2a_j^2(\mathrm{i} + u_j) = \frac{\mathrm{i}\omega S_j + L}{T_j},$$
$$\lim_{\theta\to 0} \varepsilon_j = 2a_j^2 u_j = \frac{L}{T_j}, \quad j = 1, 2. \tag{16}$$

In this case the mathematical model (1a)–(1c), (2b)–(2d) is simplified into the problem considered by Li et al. [13] and Jeng et al. [6]. Substituting (16) into (6a) leads to the solution presented by Jeng et al. [6]. On the other hand, because the specific yield S_1 of the unconfined aquifer is much greater than the confined aquifer's storativity S_2, hence $B_1^2 \neq B_2^2$. Substituting (16) into (15) leads to the approximate solution when $|\varepsilon_j| \ll |B_1^2 - B_2^2|$ $(j = 1, 2)$:

$$H_j(x,t)|_{\theta=0} = A\,\mathrm{Re}\left\{ \mathrm{e}^{\mathrm{i}\omega t}\left[\mathrm{e}^{-B_j x} - \frac{L(\mathrm{e}^{-B_j x} - \mathrm{e}^{-B_{3-j} x})}{(B_j^2 - B_{3-j}^2)T_j} \right] \right\}, \quad j = 1, 2. \tag{17}$$

Therefore, the Li et al. [13] perturbation solutions (Eqs. (12a) and (12b) in their paper) are truncated terms of (17) with respect to the perturbation parameter L/ω. Their perturbation solutions will be the same as (17) if the perturbation terms $L(h_2 - h_1)$ in Eq. (1a) and $L(h_1 - h_2)$ in Eq. (1b) of their paper are changed into only Lh_2 and Lh_1, respectively.

3.2. Comparison with Li and Jiao [10] solution

Li and Jiao [10] derived an analytical solution to investigate the effects of the leaky layer's elastic storage on the tidal groundwater head fluctuation in the lower confined aquifer. They assumed that the upper aquifer is unconfined and has a large specific yield which can damp effectively the tidal effect so that the tidal fluctuation in the upper aquifer is negligible compared to that in the lower confined aquifer. Here it will be shown that Li and Jiao's [10] solution is a special case of solution (15) and that it applies under a weaker assumption than their above-mentioned original assumption. Due to $S_1 \gg S_2$, the assumption $B_1^2 \neq B_2^2$ for solution (15) holds. Assume that the interference coefficient ε_2 in Eq. (15) corresponding to $j = 2$ is also negligible compared with $B_1^2 - B_2^2$. Substituting $\varepsilon_2 = 0$ in (15), and using (A.16) of Appendix B, yield

$$H_2(x,t) = A\,\mathrm{Re}\,(\mathrm{e}^{\mathrm{i}\omega t - B_2 x}) = A\,\mathrm{e}^{-a_2 p_2 x}\cos(\omega t - a_2 q_2 x)$$
$$\text{if } |\varepsilon_2|,\ |\varepsilon_1\varepsilon_2| \ll |B_1^2 - B_2^2|. \tag{18a}$$

Solution (18a) is the same as that of Li and Jiao [10].

Similar simplification can be made to the solution (15) corresponding to $j = 1$ if the upper aquifer's interference coefficient ε_1 is negligible. The consequent simplified solution is given by

$$H_1(x,t) = A\,\mathrm{Re}[\exp(\mathrm{i}\omega t - B_1 x)]$$
$$\text{if } |\varepsilon_1|,\ |\varepsilon_1\varepsilon_2| \ll |B_1^2 - B_2^2|. \tag{18b}$$

The conditions under which the Li and Jiao [10] solution (18a) applies are that both $\varepsilon_1\varepsilon_2$ and ε_2 can be neglected compared with $B_1^2 - B_2^2$. In order to discuss these conditions quantitatively, the ratio module of ε_j $(j = 1, 2)$ to $B_1^2 - B_2^2$ will be analyzed here. From (5) and (7a), it follows that

$$\delta_j \stackrel{\text{def}}{=} \left|\frac{\varepsilon_j}{B_2^2 - B_1^2}\right|$$
$$= 2|u_j| \Big/ \left\{ \left| \mathrm{i}\frac{\sinh[(1+\mathrm{i})\theta]}{(1+i)\theta}\left[(a_{3-j}/a_j)^2 - 1\right] + \cosh[(1+i)\theta]\left[(a_{3-j}/a_j)^2 u_{3-j} - u_j\right] \right| \right\}, \quad j = 1, 2. \tag{19}$$

Therefore, the ratio module δ_j $(j = 1, 2)$ is determined by four independent parameters: buffer capacity θ, aquifer tidal damping coefficient ratio a_1/a_2, leakage–storativity ratios u_1 and u_2. In the following it will be discussed how δ_j $(j = 1, 2)$ will change with each of the four parameters when the other three parameters are fixed.

1. *Buffer capacity θ.*
   Eq. (19) implies that

$$|\delta_j| = \mathrm{O}(\theta \mathrm{e}^{-\theta}), \quad j = 1, 2 \text{ if } \theta \gg 1. \tag{20}$$

Hence, as buffer capacity θ increases, the tidal wave interference effect between the upper and lower aquifers approaches negligible very quickly. According to (3a), either large specific storage S_S', or small vertical permeability K', or great thickness b' of the leaky layer can lead to a great buffer capacity. Here the key factor is the elastic storage of the leaky layer. If there was no elastic storage ($S_S' = 0$), then there would be no buffer capacity ($\theta = 0$). Physically, elastic storage of the leaky layer will store temporarily part of the water that flows into the leaky layer when groundwater heads in the aquifers tends to increase, then release the stored water when the groundwater heads in the aquifers tends to decrease. Hence, the elastic storage of the leaky layer acts as a buffer to weaken the interference of the groundwater head changes in the upper and lower aquifers. The speed to fill this "buffer" is proportional to the leaky layer's vertical permeability. Therefore, the buffer capacity of the leaky layer will increase with its thickness b' and specific storage S_S', and decrease with its vertical permeability K'. When the buffer capacity is great enough, the tidal wave interference between the two aquifers becomes negligible and solutions given by (18a), (18b) become applicable. Fig. 4 demonstrates how the tidal wave amplitudes in the upper and lower aquifers change with the landward distance x for different buffer capacity θ when the other four parameters a_1, a_2, u_1 and u_2 are fixed. The fixed values of the four parameters are generated by the following aquifer and tidal physical parameters: tidal amplitude $A = 1$ m, angular velocity $\omega = 4\pi\mathrm{d}^{-1}$ (semidiurnal), $T_1 = T_2 = 2400\ \mathrm{m}^2\ \mathrm{d}^{-1}$, $S_1 = 0.3$ (unconfined aquifer), $S_2 = 10^{-3}$, leakance $L = 1.0\ \mathrm{d}^{-1}$. It can been seen that with increasing of the

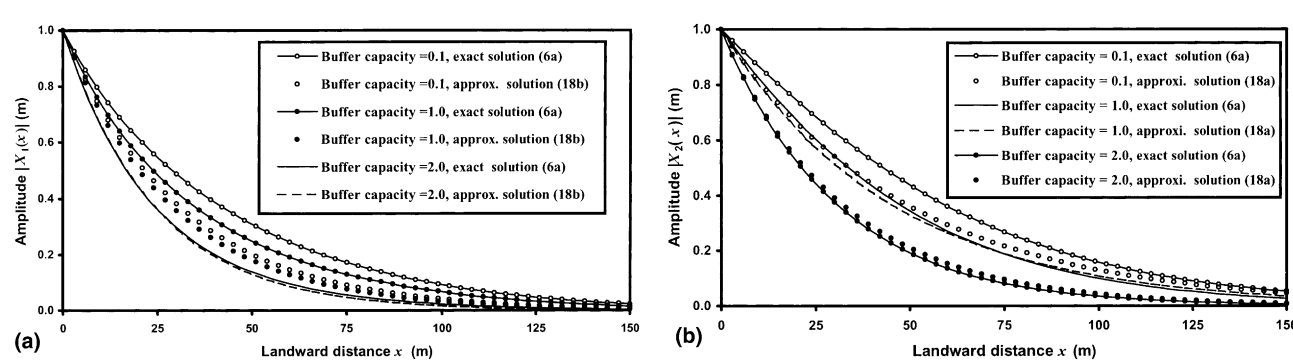

Fig. 4. Change of tidal groundwater wave amplitude (a) in the upper unconfined aquifer and (b) in the lower confined aquifer with landward distance x from the coastline for different buffer capacity θ when other parameters fixed to be $a_1 = 0.028\ \mathrm{m\ h}^{-1}, a_2 = 0.001618\ \mathrm{m\ h}^{-1}, u_1 = 0.26526, u_2 = 79.58$.

buffer capacity θ, the discrepancy between the exact solution (6a) and the approximate solution (18a), (18b) tends to zero very quickly both in the unconfined and confined aquifers. The realistic values of leakance L are usually very small and $L = 1\ \mathrm{d}^{-1}$ is unrealistically great [8]. However, an L equal to 1.0 d^{-1} is still chosen for the discussion here and below because the conclusions obtained for $L = 1\ \mathrm{d}^{-1}$ are valid for all the realistic values of L.

2. *Aquifer damping coefficient ratio* $r = a_1/a_2$.
   From (19), it follows that

$$|\delta_2| = \mathrm{O}(a_2^2/a_1^2) = \mathrm{O}\left(\frac{S_2T_1}{S_1T_2}\right),$$
$$|\delta_1| = \mathrm{O}(1) \ \text{if}\ a_1/a_2 \gg 1; \tag{21a}$$

$$|\delta_2| = \mathrm{O}(1), \quad |\delta_1| = \mathrm{O}(a_1^2/a_2^2) = \mathrm{O}\left(\frac{S_1T_2}{S_2T_1}\right)$$
$$\text{if}\ a_2/a_1 \gg 1. \tag{21b}$$

For common coastal leaky aquifer systems which satisfy $T_1/T_2 = \mathrm{O}(1)$ and $S_1/S_2 \gg 1$, the condition $a_1/a_2 \gg 1$ for (21a) holds, hence the Eq. (21a) applies. In this case, both $\varepsilon_1\varepsilon_2$ and ε_2 can be neglected compared with $B_1^2 - B_2^2$ and solution (18a) applies. Eq. (21a) is, in fact, a quantitative description and explanation of the assumption used by Jiao and Tang [7] and Li and Jiao [10]. Eq. (21b) can be discussed symmetrically for aquifers satisfying $a_2/a_1 \gg 1$. Fig. 5 demonstrates how the tidal wave amplitude in the lower confined aquifer changes with the dimensionless landward distance a_2x for different aquifer damping coefficient ratio $r = a_1/a_2$ when the other three parameters θ, u_1 and u_2 are fixed. The fixed values of the three parameters are generated by the following aquifer and tidal physical parameters: tidal amplitude $A = 1$, angular velocity $\omega = 4\pi\ \mathrm{d}^{-1}$ (semidiurnal), $S_1 = 0.3$ (unconfined aquifer), $S_2 = 10^{-3}$, leakance $L = 1.0\ \mathrm{d}^{-1}$, storativity of the leaky layer $S_S'b' = 2.5 \times 10^{-3}/\pi$. One can see that with increasing of $r = a_1/a_2$, the discrepancy between the exact solution (6a) and the approximate solution (18a) tends to zero in the confined aquifers.

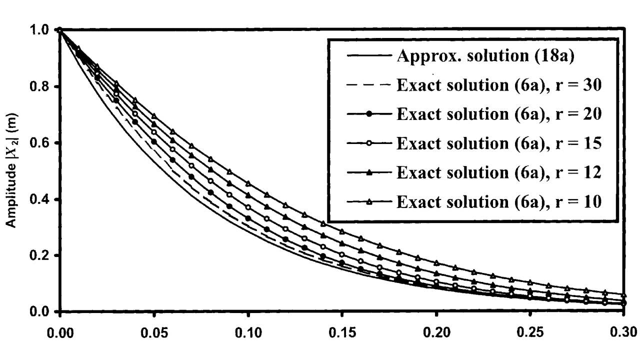

Fig. 5. Change of tidal groundwater wave amplitude in the lower confined aquifer with landward dimensionless distance a_2x for different aquifer damping coefficient ratio $r = a_1/a_2$ when other parameters fixed to be $\theta = 0.1$, $u_1 = 0.26526$, $u_2 = 79.58$.

3. *Leakage–storativity ratios* u_1 *and* u_2.
   From (19), one has

$$|\delta_2| = \mathrm{O}(u_2), \quad |\delta_1| = \mathrm{O}(u_1) = \frac{S_2}{S_1}\mathrm{O}(u_2)$$
$$\text{if}\ u_2 \ll 1; \tag{22a}$$

$$|\delta_2| = \mathrm{O}(u_2) = \frac{S_1}{S_2}\mathrm{O}(u_1),$$
$$|\delta_1| = \mathrm{O}(u_1) \ \text{if}\ u_1 \ll 1. \tag{22b}$$

Although large aquifer storativities will lead to relatively small leakage–storativity ratios u_1 or u_2, these effects are limited because a confined aquifer's storativity is small and an unconfined aquifer's storativity has a narrow range. On the other hand, small leakance L, or equivalently, small vertical permeability K' and/or great thickness b' of the leaky layer, will result in small u_1 or u_2. This situation has been discussed by Li et al. [13] and Jeng et al. [6] under the additional assumption $S_S' = 0$ (or equivalently $\theta = 0$). When $S_S' > 0$, the small value of $L = K'/b'$ will also lead to a large value of the buffer capacity $\theta = b'\sqrt{\omega S_S'/(2K')}$. Hence, according to (20), the tidal wave interference in the upper and lower aquifers will become much weaker than what is indicated by (22a), (22b).

In summary, either sufficiently large specific storage S_S', or sufficiently small vertical permeability K', or sufficiently great thickness b' of the leaky layer, can significantly weaken the tidal wave interference between the upper and lower aquifers so that the simplified solutions (18a) and (18b) apply. Sufficiently great aquifer damping coefficient ratio a_1/a_2 will also lead to negligible influence of the groundwater level variation in the upper aquifer on that in the lower aquifer so that (18a) applies. Symmetrically, sufficiently large aquifer damping coefficient ratio a_2/a_1 will lead to negligible influence of the groundwater level variation in the lower aquifer on that in the upper aquifer so that (18b) applies.

4. Conclusions

This paper presents a complete analytical solution to describe tidal groundwater level fluctuations in a coastal subsurface system. The system consists of two aquifers separated by a leaky layer. Previous solutions of Jacob [5], Jiao and Tang [7], Li and Jiao [10], Li et al. [13] and Jeng et al. [6] are all special cases of the complete solution. The solution improves the previous work in that both the effects of the leaky layer's elastic storage and the tidal wave interference between the two aquifers are considered. If the upper and lower aquifers have the same storativities and transimissivities, the system can be simplified into an equivalent aquifer–aquitard double-layered system bounded by impermeable layers from up and down. The analysis reveals that the leaky layer's elastic storage behaves as a buffer to the tidal wave interference between the upper and lower aquifers. The buffer capacity increases with the leaky layer's thickness, specific storage, and decreases with the leaky layer's vertical permeability. The tidal wave interference between the two aquifers declines quickly with the increase of the buffer capacity. Assumptions under which the Li and Jiao [10] solution applies are relaxed.

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Appendix A. Derivation of the solution (6a)

Assume that

$$H_j(x,t) = A\,\mathrm{Re}[X_j(x)\exp(\mathrm{i}\omega t)], \quad j = 1,2, \tag{A.1}$$

$$h(x,z,t) = A\,\mathrm{Re}[Z(z;x)\exp(\mathrm{i}\omega t)], \tag{A.2}$$

where $X_j(x)$ $(j = 1,2)$ is complex function of x, $Z(z;x)$ is complex function of z with x being regarded as a parameter. Substituting (A.1), (A.2) into (1a)–(1c), (2b)–(2d), and extending the nine resultant real equations into complex ones with respect to the three unknown complex functions $X_j(x)$ $(j = 1,2)$ and $Z(z;x)$, yield

$$\mathrm{i}\omega S_j X_j(x) = T_j X_j''(x) \pm K'Z'(Z_j;x), \quad 0 < x < \infty, \quad j = 2,1, \tag{A.3}$$

$$X_j(0) = 1, \quad X_j'(\infty) = 0, \quad j = 1,2; \tag{A.4}$$

$$\mathrm{i}\omega S_S' Z(z;x) = K'Z''(z;x), \quad 0 < z < b', \tag{A.5}$$

$$Z(b';x) = X_1(x), \quad Z(0;x) = X_2(x). \tag{A.6}$$

The solution of (A.5) and (A.6) is

$$Z(z;x) = \frac{X_1(x)\sinh[(1+\mathrm{i})\theta z/b'] + X_2(x)\sinh[(1+\mathrm{i})(1-z/b')\theta]}{\sinh[(1+\mathrm{i})\theta]}, \tag{A.7}$$

where θ is defined as (3a). Using (A.7), one obtains

$$K'Z'(b';x) = \frac{L(1+\mathrm{i})\theta}{\sinh[(1+\mathrm{i})\theta]}\{\cosh[(1+\mathrm{i})\theta]X_1(x) - X_2(x)\}, \tag{A.8}$$

$$K'Z'(0;x) = \frac{L(1+\mathrm{i})\theta}{\sinh[(1+\mathrm{i})\theta]}\{X_1(x) - \cosh[(1+\mathrm{i})\theta]X_2(x)\}. \tag{A.9}$$

Now substituting Eqs.(A.8) and (A.9) into (A.3), yields

$$\begin{bmatrix} \frac{\mathrm{d}^2}{\mathrm{d}x^2} - B_1^2 & \varepsilon_1 \\ \varepsilon_2 & \frac{\mathrm{d}^2}{\mathrm{d}x^2} - B_2^2 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0, \tag{A.10}$$

where B_j^2 $(j = 1,2)$ is given by (5), ε_j $(j = 1,2)$ is given by (7a). The eigenpolynomial of the coefficient matrix of (A.10) is

$$(\lambda^2 - B_1^2)(\lambda^2 - B_2^2) = \varepsilon_1\varepsilon_2. \tag{A.11}$$

The four eigenvalues of (A.11) are given by $\pm\lambda_1$ and $\pm\lambda_2$ with $\lambda_{1,2}$ defined as (6b) and $\mathrm{Re}\,\lambda_{1,2} \neq 0$ (see Appendix C). Assume $\lambda_1 \neq \lambda_2$, and using the standard method to solve linear ordinary system, the general solutions of (A.10) are

$$X_1(x) = c_1 \mathrm{e}^{-\lambda_1 x} + c_2 \mathrm{e}^{\lambda_1 x} + c_3 \mathrm{e}^{-\lambda_2 x} + c_4 \mathrm{e}^{\lambda_2 x}, \tag{A.12}$$

$$X_2(x) = c_1\mu_1 \mathrm{e}^{-\lambda_1 x} + c_2\mu_1 \mathrm{e}^{\lambda_1 x} + c_3\mu_2 \mathrm{e}^{-\lambda_2 x} + c_4\mu_2 \mathrm{e}^{\lambda_2 x}, \tag{A.13}$$

where c_j $(j = 1,\ldots,4)$ are arbitrary complex constants,

$$\mu_j = (B_1^2 - \lambda_j^2)/\varepsilon_1, \quad j = 1,2. \tag{A.14}$$

Due to $\mathrm{Re}\,\lambda_{1,2} \neq 0$, one can choose $\mathrm{Re}\,\lambda_{1,2} > 0$. Using the boundary conditions (A.4), the four constants in (A.12) and (A.13) can be determined to be $c_2 = c_4 = 0$, $c_1 = (1+\alpha_1)/2$, $c_3 = (1-\alpha_1)/2$, hence

$$X_j(x) = \frac{1}{2}[(1+\alpha_j)\exp(-\lambda_j x) + (1-\alpha_j)\exp(-\lambda_{3-j}x)], \quad j = 1,2, \tag{A.15}$$

where α_j $(j = 1,2)$ is given by (6c). Substituting (A.15) into (A.1), one obtains the solutions $H_j(x,t)$ $(j = 1,2)$ defined as (6a). Substituting (A.15) into (A.7) to obtain $Z(z;x)$, then substituting $Z(z;x)$ into (A.2), one obtains the expression of $h(x,z,t)$.

Appendix B. Proof of Re $B_{1,2} > 0$

Let

$$B_j = a_j(p_j + \mathrm{i}q_j), \quad j = 1,2. \tag{A.16}$$

Then based on (5), and using the complex equality

$$\sqrt{2(\beta + \mathrm{i}\alpha)} = \sqrt{\sqrt{\alpha^2+\beta^2} + \beta} + \mathrm{i}\sqrt{\sqrt{\alpha^2+\beta^2} - \beta} \quad \alpha, \beta > 0 \tag{A.17}$$

the four dimensionless parameters p_j and q_j $(j = 1,2)$ can be expressed as

$$p_j = p(u_j, \theta), \quad q_j = q(u_j, \theta), \quad j = 1, 2, \tag{A.18}$$

where the two functions $p(u, \theta)$ and $q(u, \theta)$ are defined as

$$p(u, \theta) = \sqrt{\sqrt{(1 + u\Theta_{\rm im}(\theta))^2 + u^2\Theta_{\rm re}^2(\theta)} + u\Theta_{\rm re}(\theta)}, \tag{A.19}$$

$$q(u, \theta) = \sqrt{\sqrt{(1 + u\Theta_{\rm im}(\theta))^2 + u^2\Theta_{\rm re}^2(\theta)} - u\Theta_{\rm re}(\theta)}, \tag{A.20}$$

in which

$$\Theta_{\rm re}(\theta) = {\rm Re}\{(1+{\rm i})\theta \coth[(1+{\rm i})\theta\} = \theta \frac{1 + 2{\rm e}^{-2\theta} \sin 2\theta - {\rm e}^{-4\theta}}{1 - 2{\rm e}^{-2\theta}\cos(2\theta) + {\rm e}^{-4\theta}}, \tag{A.21}$$

$$\Theta_{\rm im}(\theta) = {\rm Im}\{(1+{\rm i})\theta \coth[(1+{\rm i})\theta\} = \theta \frac{1 - 2{\rm e}^{-2\theta} \sin 2\theta - {\rm e}^{-4\theta}}{1 - 2{\rm e}^{-2\theta}\cos(2\theta) + {\rm e}^{-4\theta}}. \tag{A.22}$$

From (A.19), one knows that $p_j = p(u_j, \theta) > 0$, ($j = 1, 2$). Therefore, ${\rm Re} B_{1,2} > 0$.

Appendix C. Proof of Re $\lambda_{1,2} \neq 0$

If ${\rm Re}\,\lambda_{1,2} \neq 0$ does not hold, then at least one of them, say λ_1, must be a purely imaginary number. Hence one has $\lambda_1 = {\rm i}r$ with r being a real number, substituting it into (A.11), yields

$$(r^2 + B_1^2)(r^2 + B_2^2) = \varepsilon_1\varepsilon_2. \tag{A.23}$$

Using (5), (A.21) and (A.22), one obtains

$$r^2 + B_j^2 = r^2 + 2a_j^2 u_j \Theta_{\rm re}(\theta) + {\rm i}2a_j^2(u_j\Theta_{\rm im}(\theta) + 1), \quad j = 1, 2. \tag{A.24}$$

Because $r^2 \geqslant 0$, $a_j^2 > 0$, $\Theta_{\rm im}(\theta) > 0$, $\Theta_{\rm re}(\theta) > 1 > 0$ for any $\theta > 0$, and $\lim_{\theta\to 0} \Theta_{\rm re}(\theta) = 1$, $\lim_{\theta \to 0} \Theta_{\rm im}(\theta) = 0$, from (A.24), it follows that

$$|r^2 + B_j^2| \geqslant |2a_j^2 u_j \Theta_{\rm re} + {\rm i}2a_j^2(u_j\Theta_{\rm im} + 1)| > |2a_j^2 u_j(\Theta_{\rm re} + {\rm i}\Theta_{\rm im})|, \quad j = 1, 2. \tag{A.25}$$

Using the inequality $|\cosh[(1+{\rm i})\theta]| \geqslant 1$ (for any $\theta \geqslant 0$), in view of (A.25), (A.21) and (A.22), one obtains

$$|r^2 + B_j^2| > 2a_j^2 u_j |(1+{\rm i})\theta \coth[(1+{\rm i})\theta]| \geqslant 2a_j^2 u_j \left|\frac{(1+{\rm i})\theta}{\sinh[(1+{\rm i})\theta]}\right|, \quad j = 1, 2. \tag{A.26}$$

Using (A.26) and (7a), it is found that

$$|(r^2 + B_1^2)(r^2 + B_2^2)| > 4a_1^2 u_1 a_2^2 u_2 \left|\frac{(1+{\rm i})\theta}{\sinh[(1+{\rm i})\theta]}\right|^2 = |\varepsilon_1\varepsilon_2|. \tag{A.27}$$

Inequality (A.27) is in contradiction with Eq. (A.23). Therefore, ${\rm Re}\,\lambda_{1,2} \neq 0$.

Appendix D. Justification of neglecting the term $^2h/x^2$ in (2a)

In order to show that $\partial^2 h/\partial x^2$ is negligible compared with $\partial^2 h/\partial z^2$, based on the transform (A.2), it is adequate if one can show that $\partial^2 Z/\partial x^2$ is negligible compared with $\partial^2 Z/\partial z^2$ near the coastline where the functions $X_j(x)$ ($j = 1, 2$) change with x sharply. Using the notation

$$f(z/b') = \frac{\sinh[(1+{\rm i})\theta z/b']}{\sinh[(1+{\rm i})\theta]}, \tag{A.28}$$

from (A.5) and (A.7), one has

$$\frac{\partial^2 Z}{\partial z^2} = \frac{{\rm i}\omega S_S'}{K'}[X_1 f(z/b') + X_2 f(1 - z/b')]. \tag{A.29}$$

Using (A.7), one has

$$\frac{\partial^2 Z}{\partial x^2} = X_1''(x) f(z/b') + X_2''(x) f(1 - z/b'). \tag{A.30}$$

Near the coastline where the functions $X_j(x)$ ($j = 1, 2$) change with x sharply, due to the boundary condition $X_j(0) = 1$ ($j = 1, 2$), one has

$$|X_j(x)| = {\rm O}(1) \quad (j = 1, 2). \tag{A.31}$$

If one can show that

$$|X_j''(x)| \ll \frac{\omega S_S'}{K'}|X_j(x)| \quad (j = 1, 2), \tag{A.32}$$

under the condition (A.31), then from (A.30) and (A.29), it immediately follows that $\partial^2 Z/\partial x^2$ is negligible compared with $\partial^2 Z/\partial z^2$. Using (A.3), (A.8) and (A.9), one has

$$X_1'' = \frac{{\rm i}\omega S_1 X_1}{T_1} + \frac{K'Z'(b'; x)}{T_1} = \frac{{\rm i}\omega S_1 + L(1+{\rm i})\theta\coth[(1+{\rm i})\theta]}{T_1}X_1 - \frac{L(1+{\rm i})\theta}{T_1 \sinh[(1+{\rm i})\theta]}X_2, \tag{A.33}$$

$$X_2'' = \frac{{\rm i}\omega S_2 X_2}{T_2} - \frac{K'Z'(0; x)}{T_2} = \frac{{\rm i}\omega S_2 + L(1+{\rm i})\theta\coth[(1+{\rm i})\theta]}{T_2}X_2 - \frac{L(1+{\rm i})\theta}{T_2 \sinh[(1+{\rm i})\theta]}X_1. \tag{A.34}$$

In the following only the case of $j = 1$ for (A.32) will be elaborated. The case of $j = 2$ is similar and will be omitted. Under condition (A.31), one has

$$|X_1''| \leqslant \left(\frac{\omega S_1}{T_1} + \frac{L}{T_1} \left| \frac{(1+\mathrm{i})\theta}{\sinh[(1+\mathrm{i})\theta]} \right| + \frac{L}{T_1} |(1+\mathrm{i})\theta \coth[(1+\mathrm{i})\theta]| \right) \mathrm{O}(1). \tag{A.35}$$

Based on (2e), one has

$$\frac{\omega S_1}{T_1} \ll \frac{\omega S_S'}{K'}, \quad \frac{L}{T_1} \left| \frac{(1+\mathrm{i})\theta}{\sinh[(1+\mathrm{i})\theta]} \right| \leqslant \frac{L}{T_1} \mathrm{O}(1) \ll \frac{\omega S_S'}{K'}. \tag{A.36}$$

Here the estimation

$$\left| \frac{(1+\mathrm{i})\theta}{\sinh[(1+\mathrm{i})\theta]} \right| = \begin{cases} \mathrm{O}(1) & \text{if } \theta \leqslant \mathrm{O}(1), \\ \mathrm{O}(1) & \text{if } \theta \gg 1. \end{cases} \tag{A.37}$$

is used. For small values of θ ($\theta \leqslant \mathrm{O}(1)$), one has

$$|(1+\mathrm{i})\theta \coth[(1+\mathrm{i})\theta]| = \mathrm{O}(1) \quad \text{if } \theta \leqslant \mathrm{O}(1). \tag{A.38}$$

Based on (2f) and (A.38), it follows that

$$\frac{L}{T_1} |(1+\mathrm{i})\theta \coth[(1+\mathrm{i})\theta]| \ll \frac{\omega S_S'}{K'} \quad \text{if } \theta \leqslant \mathrm{O}(1). \tag{A.39}$$

For great values of $\theta (\theta \gg 1)$, one has

$$|(1+\mathrm{i})\theta \coth[(1+\mathrm{i})\theta]| = \theta \mathrm{O}(1) \quad \text{if } \theta \gg 1. \tag{A.40}$$

In this case, using the definition Eq. (3a) of θ, one can easily check that

$$\frac{L}{T_1} |(1+\mathrm{i})\theta \coth[(1+\mathrm{i})\theta]| \ll \frac{\omega S_S'}{K'} \mathrm{O}(1) \quad \text{if } \theta \gg 1 \tag{A.41}$$

is equivalent to

$$\frac{T_1 \theta}{T'} \gg \mathrm{O}(1) \quad \text{if } \theta \gg 1, \tag{A.42}$$

which is obvious. Therefore, (A.41) is valid. Combining (A.36) with (A.39) and (A.41), one proves (A.32) for the case of $j = 1$.

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