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Advances in Water Resources 27 (2004) 981–990

Advances in Water Resources

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Modified Theis equation by considering the bending effect of the confining unit

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Received 1 September 2003; received in revised form 30 June 2004; accepted 3 August 2004

Abstract

The Theis equation has been widely used to study the transient movement of groundwater as a result of pumping in a confined aquifer. It is well known that the observed drawdown at early times has an obvious departure from the theoretical drawdown based on the Theis equation. The Theis equation was derived under the assumption that total stress in the aquifer was constant and the mechanical behavior of the confining unit was neglected. However, most geological formations, especially those which are well consolidated, have rigidity and therefore may bend like a plate to a certain extent. The increase in the effective stress in the aquifer due to pumping may not contribute entirely to the compression of the aquifer, but may be partially cancelled out by bending of the overlying aquitard. This means only a part of the total stress is used to compact the aquifer, or the aquifer cannot produce as much water as estimated from the Theis equation. This paper investigated the impact of the bending effect of the confining unit on drawdown. An analytical model which couples flow in the aquifer and bending of the confining unit was presented. The theory is based on elastic plates and solutions were given to the drawdown of groundwater level and deflection of the overlying formation. The drawdown estimated from the new equation was compared with that from the Theis equation. It can be concluded that drawdown from the Theis equation is less than the drawdown predicted by including the bending effect of the confining unit. Both a hypothetical example and a field pumping test in Shandong Province, China, were used to demonstrate the bending effect of the confining unit in the analysis of pumping test data. This paper demonstrated that the initial disagreement between observed drawdown and the Theis solution could be caused by the bending effect of the confining unit, a phenomenon not well addressed in traditional pumping test analysis. A quantitative understanding of this phenomenon can provide improved guidelines for analyzing drawdown data in a confined aquifer.

Keywords: Confined aquifer; Pumping test; Theis equation; Modified Theis equation; Bending effect

1. Introduction

Groundwater flow induced by pumping is one of the fundamental issues in groundwater hydraulics. It is well known that the first solution of transient drawdown around a pumping well was given by Theis [15]. It was directly obtained from an analogous solution in heat conduction theory. The solution was further vindicated by Jacob [7]. Since then, the Theis equation has been widely used in the groundwater community [4]. Under the assumption that a fully penetrating well pumps water from an isotropic, homogenous, and infinite confined aquifer without inflow from surrounding formations, Theis's solution gives the drawdown of water level, s, at a radial distance, r, and at the time, t, as

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doi:10.1016/j.advwatres.2004.08.007

$$s = \frac{Q}{4\pi T}\int_u^\infty \frac{e^{-y}}{y}\,dy \tag{1}$$

Usually (1) is written in the form

$$s = \frac{Q}{4\pi T}W(u) \tag{2}$$

where Q is pumping rate, T is transmissivity, $u = r^2S/4Tt$ is a dimensionless group parameter, S is storage coefficient (dimensionless) and $W(u)$ is exponential integral.

The formula (2) has been widely used to analyze pumping-test data to determine the parameters S and T of aquifers and subsequently modified for semi-confined aquifers by including leakage and compression of the aquitard (e.g., [8,9]).

For most pumping tests, during the early time the observed drawdown is larger than that calculated from the Theis equation. Such a difference is believed to be caused by the difference between the real aquifer conditions and the assumptions behind the Theis equation. As pointed out by Kruseman and de Ridder [12], the theoretical equation is based on the assumptions that the well discharge remains constant and that the release of the water stored in the aquifer is immediate and directly proportional to the rate of decline of the pressure head. In fact, there may be a time lag between the pressure decline and the release of stored water, and initially the well discharge may vary as the pump is adjusting itself to the changing head. Some model the difference as being caused by well storage (e.g., [5,14]).

The difference may be caused by the mechanical reaction of the confining unit in response to pumping. The mechanical relation between the confined aquifer and its confining unit was not mentioned by Theis [15]. This relation was discussed briefly by Jacob [7] who assumed that the total stress (summation of effective stress and pore pressure) in the aquifer remained constant while the piezometric surface of groundwater changes. A portion of the pumped water results from water expansion and aquifer compression in response to an increase in effective stress caused by pumping.

With compression of the confined aquifer due to groundwater lowering, the confining unit moves down and nonuniform subsidence occurs. Jacob's assumption implied that the movement of the confining unit is controlled by the cone of depression and the loading above the aquifer and that the confining unit moves like a very soft plate, or a series of loosely connected blocks with no mechanical force between the blocks (Fig. 1(a)).

However, most geological formations, especially those which are well consolidated, have rigidity and therefore may behave like a plate to a certain extent (this is denoted as bending effect of the confining unit in this paper). The confining layer will not bend downwards as freely as, if it were, loosely connected blocks. Or, the friction at the interfaces of blocks can support the deformation to some degree (Fig. 1(b)). In consequence that the bending of the overlying formation will reduce the loading transferred to the aquifer. This bending effect was noted by Chen and Wu [3] during a pumping test in a confined aquifer. In a pumping test analysis for a particular aquifer, they speculated that the increase in the effective stress in the aquifer due to pumping may not contribute entirely to the compression of the aquifer, but may partially cancelled out by the overlying aquitard, or the plate. They believed that this bending effect could be one of the reasons responsible for larger observed drawdown than that estimated from the Theis equation.

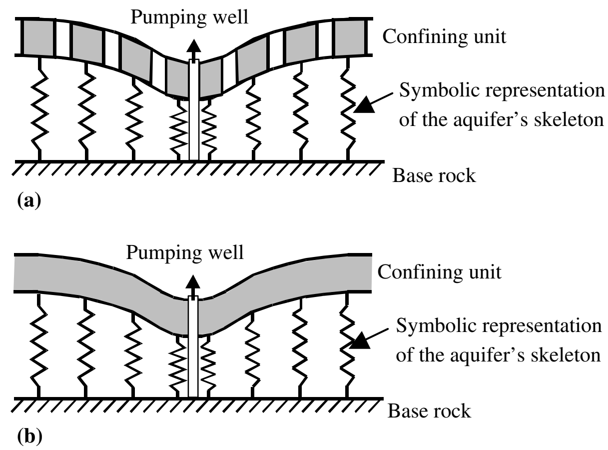

Fig. 1. Schematics for a confined aquifer-well system and subsidence of the confining unit. (a) Assumption used by Theis equation: the confining unit moves like a very soft belt, or a series of loosely connected blocks with no mechanical force between the blocks; (b) assumption used in this paper: the confining unit has rigidity and behaves like a beam to some degree.

Various interesting phenomena related to pumping in aquifer–aquitard systems have been investigated by previous researchers using approaches beyond standard groundwater theory. For example, to explain the so-called Noordbergum effect (the rising of hydraulic head in the overlying or underlying aquitard in response to pumping in the main aquifer), both analytical and numerical studies which couple fluid flow with aquifer deformation based on Biot's theory of consolidation were developed to examine the water level fluctuation in the aquitard during pumping (e.g., [11,16]). Helm [10] presented analytical solutions to investigate the horizontal aquifer movement in response to pumping. A comprehensive review of various studies related to aquifer tests is beyond the scope of this paper and interested readers can find such a good review of such studies in Kruseman and de Ridder [12], Kim and Parizek [11], and Helm [10].

For a real pumping test in a complicated multiplayer aquifer system, various effects such as the Noordbergum effect, horizontal aquifer movement, and the bending effect will interact with each other and influence the drawdown in the pumped aquifer. This paper will focus only on the bending effect of the confining unit and how it will modify the drawdown calculated by the Theis equation. Instead of using a complicated poroelastic solid theory or Biot's theory of three-dimensional consolidation, this paper employs a much simpler approach based on the theory of thin elastic plates to investigate the bending effect of the confining unit. An analytical solution of the same format as the classic Theis equation will be presented for drawdown of groundwater level in the aquifer. It will be demonstrated that Theis's solution is a special case in which the confining unit has no rigidity or the aquifer matrix is incompressible. The drawdown from the Theis equation is less than that from the modified Theis equation which includes the bending effect of the confining unit. The difference increases when the radial distance is small and time is short. The findings of this paper can provide additional insights in understanding the difference between observed drawdown and the drawdown based on the traditional Theis equation when analyzing early drawdown data in a confined aquifer.

2. Stress analysis

Consider a horizontal confined aquifer extending towards infinity under a confining unit (Fig. 2). Water is extracted by a fully penetrating well. The total stress σ, the effective stress σ' and the pore pressure p in the aquifer have a relation presented by the principle of effective stress (e.g., Bear [2])

$$\sigma = \sigma' + p \tag{3}$$

Eq. (3) describes the mechanical relationship of groundwater and aquifer medium. This equation assumes that the aquifer is saturated or almost saturated by water and the soil grains are assumed incompressible, which is the case for the aquifer system in this study.

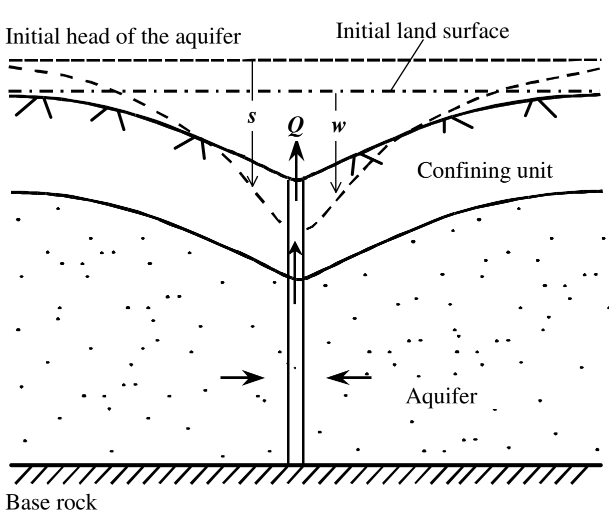

Fig. 2. Idealized representation of the well-flow system for a confined aquifer. Arrows indicate the water flow directions. s is the drawdown of the aquifer and w is the deflection of confining unit. Q is the pumping rate.

In addition to the variations of effective stress $\Delta\sigma'$ and pore pressure Δp resulting from groundwater lowering, the total stress $\Delta\sigma$ changes with the bending of the confining formation. The changes in effective stress, pore pressure and total stress have the following relation:

$$\Delta\sigma = \Delta\sigma' + \Delta p \tag{4}$$

On the assumption that the deformation is elastic and occurs only in the vertical direction, the compression of the confined aquifer is presented as following

$$\Delta b/b = \alpha\Delta\sigma' \tag{5}$$

where b is the thickness of the aquifer and Δb is its reduction due to increasing of the effective stress, α is the compressibility of the elastic aquifer matrix.

Assume that the deformation of base rock (the formation below the aquifer) is zero. The reduction of the aquifer thickness is then equal to the deflection (vertical downward displacement, w) of the confining layer:

$$w = \Delta b = b\alpha\Delta\sigma' \tag{6}$$

The change of pore pressure associated with drawdown s is

$$\Delta p = -\gamma_w s \tag{7}$$

where γ_w is the specific weight of water. The volume of water released from a unit horizontal area of the aquifer, ΔV, due to the compaction of the aquifer skeleton and expansion of water can be expressed as:

$$\Delta V = w - nb\beta\Delta p = b(\alpha\Delta\sigma' + n\beta\gamma_w s) \tag{8}$$

where β is the compressibility of water and n is the porosity of the aquifer. The compressibility of water is defined as relative increment of its density with a unit increment of pore pressure while temperature is constant.

In conventional groundwater theory σ is assumed to be constant. Then $\Delta\sigma' = \gamma_w s$. Denote the volume of water released from a unit horizontal area of the aquifer as $\Delta V'$, then

$$\Delta V' = \gamma_w b(\alpha + n\beta)s = (\mu_m + \mu_w)s \tag{9}$$

In this study μ_w and μ_m are defined as storage coefficients of water expansion and skeleton compression within the aquifer, respectively:

$$\mu_m = \gamma_w b\alpha, \quad \mu_w = \gamma_w bn\beta \tag{10}$$

For a confined aquifer the total storage coefficient is

$$S = \mu_w + \mu_m \tag{11}$$

Taking the bending effect into consideration, we release the total stress σ from constant. From (4) and (8) we have

$$\Delta\sigma' = \Delta\sigma - \Delta p \tag{12}$$

$$\Delta V = \Delta V' + b\alpha\Delta\sigma \tag{13}$$

where ΔV is associated with the variation of total stress σ. Since a force is produced by the bending of the confining layer, which reduces the total stress transferred to the aquifer, we have $\Delta\sigma < 0$, and $\Delta V < \Delta V'$. It indicates that, due to the bending effect of the overlying formation, the aquifer will not be compressed or the groundwater will not be released as much as traditionally calculated. Therefore, the actual drawdown is greater than what is estimated by the Theis equation.

3. Model and solutions

For the aquifer well system showed in Fig. 2, the transient radial flow of the confined aquifer can be described with the following equation

$$T\left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r}\right) = \frac{\partial}{\partial t}(\Delta V) + q(r,t) \tag{14}$$

where q is the sink or source term. According to (8) and (10), (14) can be rewritten as

$$T\left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r}\right) = \frac{\partial w}{\partial t} + \mu_{\mathrm{w}}\frac{\partial s}{\partial t} + q(r,t) \tag{15}$$

Considering 6,7, and 10,4 can be rearranged into

$$\Delta\sigma = \frac{\gamma_{\mathrm{w}}}{\mu_m}w - \gamma_{\mathrm{w}}s \tag{16}$$

The change of total stress in the aquifer is also the additional loading which causes the bending of the confining unit. According to the theory of thin plates on small deflection (see Arthur and Omar [1]), the equation governing the bending of the confining unit is

$$D\nabla^4 w = -\Delta\sigma = -\frac{\gamma_{\mathrm{w}}}{\mu_m}w + \gamma_{\mathrm{w}}s \tag{17}$$

where D is the flexural rigidity of the overlying layer of the confined aquifer and

$$D = \frac{Eb_1^3}{12(1-v^2)} \tag{18}$$

where E and b_1 are the Young's modulus and thickness of the overlying layer respectively, v is its Poisson's ratio.

The actual characteristics of the layered mediums in the overlying formation are complex. In order to avoid the complexity in mathematical description, features such as the plasticity and rheologicity of the formation are ignored. It is assumed that the confining unit is not very thick related to its large areal extent and can be treated approximately as a thin plate, so that the theory of thin plates can be utilized.

With initial and boundary conditions, the mathematical model integrating well flow and the bending effect of the overlying formation is given as

$$D\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)w + \frac{\gamma_{\mathrm{w}}}{\mu_m}w = \gamma_{\mathrm{w}}s,$$
$$t > 0,\ r > 0 \tag{19}$$

$$T\left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r}\right) = \frac{\partial w}{\partial t} + \mu_{\mathrm{w}}\frac{\partial s}{\partial t} + q(r,t), \quad t > 0,\ r > 0 \tag{20}$$

$$\lim_{r_{\mathrm{w}}\to 0}\int_0^{r_{\mathrm{w}}} 2\pi r q(r,t)\,\mathrm{d}r = -Q,$$
$$r \leqslant r_{\mathrm{w}};\quad q(r,t) = 0,\quad r > r_{\mathrm{w}},\quad t > 0 \tag{21}$$

$$\partial w/\partial r = 0,\quad r = 0,\ t > 0 \tag{22}$$

$$s(\infty,t) = 0,\quad w(\infty,t) = 0,\quad t > 0 \tag{23}$$

$$s(r,0) = 0,\quad w(r,0) = 0,\quad r > 0 \tag{24}$$

where r_{w} is the radius of the well. The initial condition is presented in (24). The boundary conditions of drawdown and the bending deflection are shown in (22) and (23).

The solutions for the drawdown and deflection are (see Appendix A for details) respectively:

$$w(r,t) = \frac{Q}{2\pi}\int_0^\infty \frac{1-\mathrm{e}^{-\lambda(\beta)t}}{a\beta(1+c\beta^4)}\left(1 - \frac{\mu_{\mathrm{w}}}{S}\right)J_0(\beta r)\,\mathrm{d}\beta \tag{25}$$

$$s(r,t) = \frac{Q}{2\pi T}\int_0^\infty \frac{1-\mathrm{e}^{-\lambda(\beta)t}}{\beta}J_0(\beta r)\,\mathrm{d}\beta \tag{26}$$

where

$$\lambda(\beta) = a\beta^2\frac{1+c\beta^4}{1+(\mu_{\mathrm{w}}c\beta^4/S)} \tag{27}$$

$$a = T/S,\quad c = \mu_m D/\gamma_{\mathrm{w}} \tag{28}$$

and $J_0(x)$ is the Bessel function of the first kind of order zero.

With the definition of group variables

$$U = \frac{1}{u} = \frac{4at}{r^2},\quad V = \frac{c}{r^4},\quad x = (\beta r)^2 \tag{29}$$

and integral including the bending effect of the confining unit:

$$M\left(U,V,\frac{\mu_{\mathrm{w}}}{S}\right) = \int_0^\infty \left[1 - \mathrm{e}^{-(Ux/4)(1+Vx^2)/(1+\mu_{\mathrm{w}}Vx^2/S)}\right] \times \frac{J_0(\sqrt{x})}{x}\,\mathrm{d}x, \tag{30}$$

(26) can be written as

$$s = \frac{Q}{4\pi T}M\left(U,V,\frac{\mu_{\mathrm{w}}}{S}\right) = \frac{Q}{4\pi T}M\left(\frac{4at}{r^2},\frac{c}{r^4},\frac{\mu_{\mathrm{w}}}{S}\right) \tag{31}$$

In Appendix B it is demonstrated that

$$\text{when } c = 0, \quad M\left(\frac{1}{u}, 0, \frac{\mu_w}{S}\right) = W(u) \tag{32}$$

$$\text{when } c = \infty, \quad M\left(\frac{1}{u}, \infty, \frac{\mu_w}{S}\right) = W\left(\frac{\mu_w r^2}{4Tt}\right) \tag{33}$$

Eq. (32) means that Theis's solution is a special case when the overlying formation has no rigidity ($D = 0$), or when the aquifer medium is incompressible ($\mu_m = 0$) so that only the compressibility of water provides the water being pumped. $c = \infty$ in (33) means the flexural rigidity, D, of the confining unit is extremely great and bending does not occur.

4. Comparison of the modified Theis solution with the Theis solution

The preceding analysis shows that the drawdown is a function of three dimensionless parameters: $4at/r^2$, r^4/c, μ_w/S. For convenience of the following discussion, more dimensionless group parameters are defined as follows

$$s_D = s/(Q/4\pi T)$$
$$t_D = 4at/r^2$$
$$r_D = r^4/c$$

Fig. 3 shows plots of s_D versus t_D for a given μ_w/S when $r_D = 1$ and 0.01. The curve from Theis' solution, which is equivalent to the case when $r_D = \infty$, is also presented in the figure. It can be seen that the dimensionless drawdown from the Theis equation is always less than that affected by the bending effect of the confining unit. The deviation decreases rapidly with dimensionless radial distance r_D, which also indicates that the error from the Theis equation will be large near the pumping well.

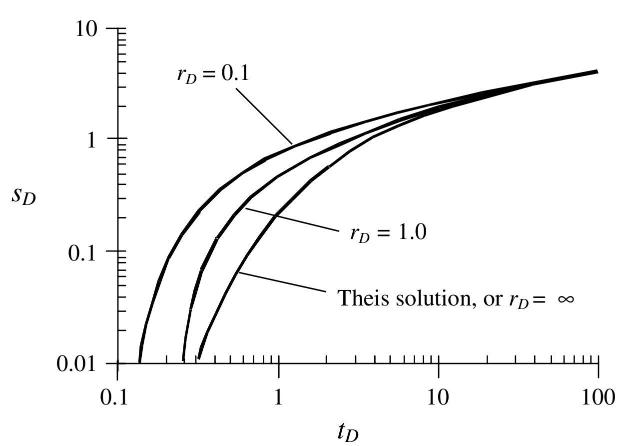

Fig. 3. Change of dimensionless drawdown s_D with dimensionless time t_D for various values of r_D while $\mu_w/S = 0.2$.

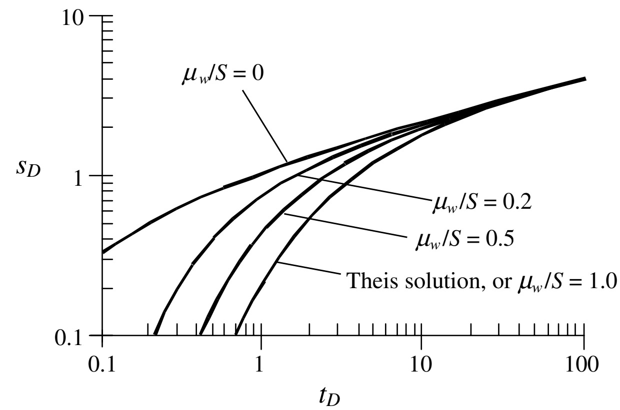

Fig. 4. Change of dimensionless drawdown s_D with dimensionless time t_D for various values of μ_w/S while $r_D = 0.1$.

The influence of μ_w/S on time-drawdown curves is shown in Fig. 4. As can be seen from the definitions of μ_w, μ_m and S, the parameter μ_w/S ranges from 0 to 1. If the aquifer skeleton is incompressible, then $\mu_w/S = 1$. In the case of $\mu_w/S = 0.5$, half of the released storage is contributed from water expansion. For a given value of r_D, the Theis equation is represented by a curve independent of μ_w/S, but a time-drawdown curve determined by the modified Theis equation (31) depends on μ_w/S. Only for $\mu_w/S = 1$ does the curve from the modified equation approach that from the Theis equation. In the cases of $\mu_w/S < 1$, Fig. 4 shows that the bending effect of the confining formation will lead to an increase in the drawdown in the early time relative to Theis solution. At a large value of t_D, however, drawdown on basis of (31) becomes very close to Theis solution.

5. Case studies on the bending effect

For the convenience of discussion in the previous section, dimensionless time and drawdown are used (Figs. 3 and 4), but the physical implication of these dimensionless numbers is not always straightforward. A hypothetical example and a real pumping test are chosen to discuss further the significance of the bending effect.

The impact of the bending effect on the estimated parameters will be investigated when the bending effect is significant but ignored. Drawdowns will be created using the modified Theis equation and 'true' aquifer parameters. The observed drawdowns are then fitted by the Theis equation to estimate the aquifer parameters. By comparing the true and the estimated parameters, the impact of the bending effect can be analyzed.

An optimization method based on simplex optimization algorithm is used to identify these parameters. An introduction of simplex method for parameter estimation in groundwater flow problems can be found in

Devlin [6]. An error function is used as a measure of the discrepancy between the observed and calculated drawdowns. This function is defined as

$$\Gamma(T, a, c, \mu_w/S) = \sum_{i=1}^{N} \left(\log \frac{s'_i}{s_i} \right)^2 \tag{34}$$

where s and s' are observed and calculated drawdowns, respectively; N is the total number of drawdowns. Optimized parameters are subjected to the least value of the error function.

5.1. Example 1: a hypothetical aquifer system

The hypothetical aquifer is a two-layer system consisting of a silty sand aquifer confined by a clayey silt aquitard. The layers extend infinitely horizontally and the mechanical and hydraulic properties of each layer are homogeneous. The properties of the aquifer system are specified in Table 1. The aquifer is pumped at a rate of 1000 m^3/d from a fully penetrating well. In order to focus on the bending effect, the confining unit is assumed to be completely impervious and leakage from this layer is ignored.

Assume that there are two observation wells in the aquifer, with radial distances $r = 10$ m and $r = 20$ m respectively from the pumping well. Fig. 5 shows how the drawdown in the observation wells changes with time. Due to the bending effect, the difference in drawdowns at $r = 10$ m between the Theis solution and the modified Theis equation is 27 cm in the first two minutes and then gradually decreases with time. However, at $r = 20$ m, the difference is less than 10 cm and the time when the maximum difference is achieved is 8 min after the pumping starts. The bending effect is weak while the distance is greater than 20 m. This example demonstrates that the bending effect decreases with pumping time and radial distance but is significant near the pumping well in the early pumping time.

Table 1
Properties of the aquifer system in Example 1

Layer	Property	Value
Confining unit: clayey silt	Layer thickness, b_1	50.00 m
	Young's modulus, E	7.00×10^6 N/m^2
	Poisson's ratio, v	0.30
	Flexural rigidity, D	8.01×10^{10} N m
The confined aquifer: silty sand	Layer thickness, b	100.00 m
	Transmmisivity, T	200.00 m^2/d
	Total storage coefficient, S	8.00×10^{-3}
	Porosity, n	0.25
	Storage coefficient of water expansion, μ_w	0.12×10^{-3}
	Storage coefficient of skeleton compression, μ_m	7.88×10^{-3}
	Derived group parameters	$a = 0.25 \times 10^5$ m^2/d $c = 6.31 \times 10^4$ m^4 $\mu_w/S = 1.50\%$

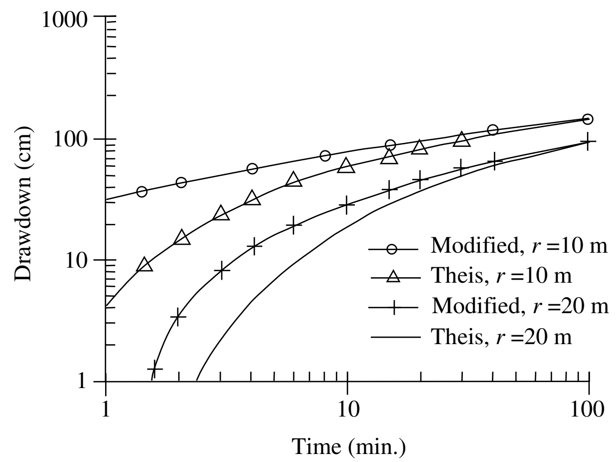

Fig. 5. Change of drawdowns with time at $r = 10$ m and 20 m obtained from modified equation and Theis equation, in Example 1.

To understand how the bending effect may impact the estimated aquifer parameters when the effect is ignored, the drawdown calculated from the modified solution is treated as observed drawdown and reanalyzed by Theis equation to estimate the parameters T and S using the simplex optimization algorithm. At the radial distance $r = 10$ m, the errors in estimated T and S can be 102% and 82%, respectively, when the drawdowns between 1 and 10 min are used (Table 2). The errors are reduced to less than 80% when the drawdowns between 1 and 100 min are used. At the radial distance $r = 20$ m, the errors in estimated T and S are much smaller. This example demonstrates that the errors in estimated aquifer parameters can be significant near the pumping well at the early pumping time.

The common practice in fitting the observed data with the Theis equation is to ignore the first few drawdowns in the early pumping time. The last two rows in Table 2 show that such an approach can reduce significantly the errors in estimated T and S.

5.2. Example 2: a real aquifer system in Shandong, China

A pumping test was conducted in a confined limestone aquifer in Xishang Village in Shandong Province, China [3]. The test lasted for about 24 h. The change of pumping rate with time is shown in Table 3. The confining unit of the aquifer is sandstone and the rock below the aquifer is diorite, which is practically impermeable. A simplified geological map around the test site, together with a cross-section, is presented in Fig. 6. The pumping well ZK95 had a radius of 0.055 m and fully penetrated through the entire aquifer thickness of 83.5 m. Detailed discussion on the geology and hydrogeology was presented by Chen and Wu [3].

Table 2
Change of estimated parameters T and S and degrees of errors with pumping time involved in Example 1 when the bending effect is ignored

Time involved (min)	$r = 10$ m				$r = 20$ m			
	T (m^2/d)	$\lvert T - T_0\rvert/T_0$	S (10^{-3})	$\lvert S - S_0\rvert/S_0$	T (m^2/d)	$\lvert T - T_0\rvert/T_0$	S (10^{-3})	$\lvert S - S_0\rvert/S_0$
1–10	403.19	1.02	1.48	0.82	257.74	0.29	4.96	0.38
1–30	377.81	0.89	1.58	0.80	272.97	0.36	5.14	0.36
1–100	349.68	0.75	1.78	0.78	271.11	0.36	5.12	0.36
10–100	304.38	0.52	2.47	0.69	254.77	0.27	5.61	0.30
30–100	224.99	0.12	5.41	0.32	219.81	0.10	6.60	0.18

The 'true' parameter values $T_0 = 200$ m^2/d and $S_0 = 8.0 \times 10^{-3}$.

Table 3
Pumping rate vs. time in Well ZK95 in Example 2

Pumping time (min)	<50	50	180	1414
Pumping rate, Q (m^3/d)	629.6	723.0	751.0	0

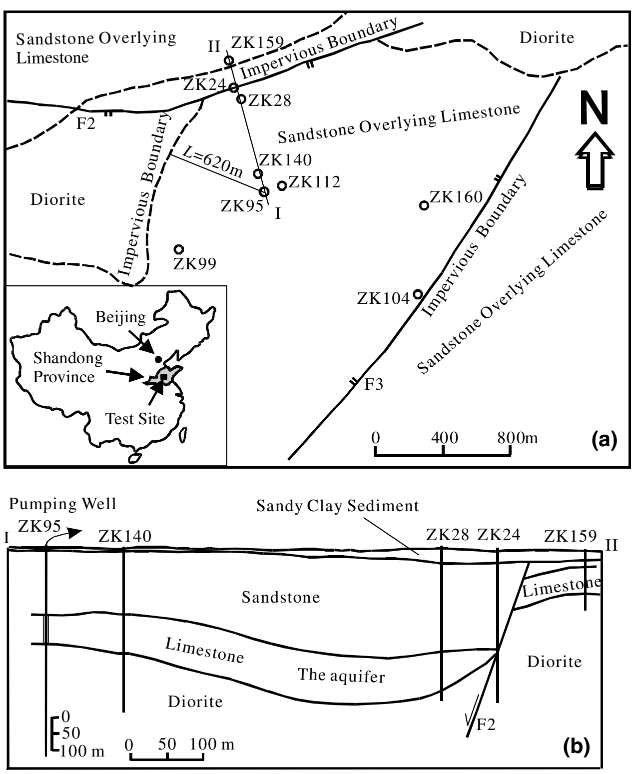

Fig. 6. The simplified geological map (a) of the pumping test site in Shandong, China (the pumping well is ZK95, with ZK112 as an observation well. F2 and F3 are faults) and the geological cross-sections I and II (b), in Example 2.

This limestone is well-jointed and corrosion can be observed along joints and fractures but there is no major karstic feature such as caves. For this pumping test, the limestone can be regarded approximately as a homogeneous porous aquifer [3]. The aquifer is bounded on the southeast and northwest by faults F3 and F2 respectively that are almost impermeable. The distance from ZK95 to the nearest boundaries is 620 m. Drawdown was observed in ZK112, a borehole 104 m from the pumping well ZK95. As discussed by Chen and Wu [3], the boundary-effect was insignificant before 432 min of pumping. Chen and Wu analyzed the drawdown data with Theis equation and estimated the aquifer parameters T and S. The match between the observed and the calculated drawdown was generally satisfactory, but the observed drawdown during the initial 50 min was found to be greater than that calculated by Theis equation and the difference was up to 3.0 cm [3].

The test data is re-analyzed using the modified Theis equation considering the bending effect. Like the Theis equation, the modified Theis equation is a linear system, so the principle of superposition can be used to handle the pumping rate that changed with time. Four parameters (T, a, c, μ_w/S) are needed to describe the drawdown changing with time (Eq. (31)). The simplex optimization algorithm is used to identify these parameters. The parameters which lead to the best fit between the observed and calculated drawdowns are: $T = 587.04$ m^2/d, $a = 5.77 \times 10^5$ m^2/d, $c = 1.36 \times 10^8$ m^4, $\mu_w/S = 3.73\%$. Among them, c is a new parameter introduced in this paper and mainly controlled by the mechanical properties of the confining unit sandstone. To provide a better idea about the physical meaning of this parameter, a set of mechanical parameters such as Young's modulus, E and Poisson's ratio, v of the sandstone which lead to $c = 1.36 \times 10^8$ m^4 are presented in Table 4. The values of these mechanical parameters are within the typical ranges of mechanical properties of sandstones presented by Lama and Vutukuri [13].

The observed drawdown at ZK112 and the results calculated from the Theis equation and the modified equation are shown in Fig. 7. As can be seen, when time is less than 100 min, the calculated drawdown from the modified Theis equation matches much better with the observed data than that from the Theis equation.

6. Summary

In the traditional theory of confined groundwater flow toward a well, the total stress in the aquifer is assumed to be constant so that drawdown is directly proportional to the increment of effective stress. The

Table 4
Properties of the aquifer-system in Example 2

Layer	Lithology	Property	Value
Confining unit	Sandstone	Thickness, b_1	160.00 m
		Young's modulus, $E^{(2)}$	3.92×10^9 N/m^2
		Poisson's ratio, $\nu^{(1)}$	0.15
		Flexural rigidity, $D^{(2)}$	1.37×10^{15} N m
Confined aquifer	Limestone	Layer thickness, b	83.54 m
		Transmmisivity, $T^{(3)}$	587.04 m^2/d
		Total storage coefficient, $S^{(2)}$	1.02×10^{-3}
		Porosity, $n^{(2)}$	0.11
		Storage coefficient of water expansion, $\mu_w^{(2)}$	0.38×10^{-4}
		Storage coefficient of skeleton compression, $\mu_m^{(2)}$	9.82×10^{-4}
Base rock	Diorite	Thickness	>300.00 m
		Estimated group parameters$^{(3)}$	$a = 5.77\times10^5$ m^2/d $c = 1.36\times10^8$ m^4 $\mu_w/S = 3.73\%$

Note: (1) values are within the ranges of typical parameters for sandstones presented by Lama & Vutukuri [13]; (2) derived from estimated group parameters; (3) optimized parameters.

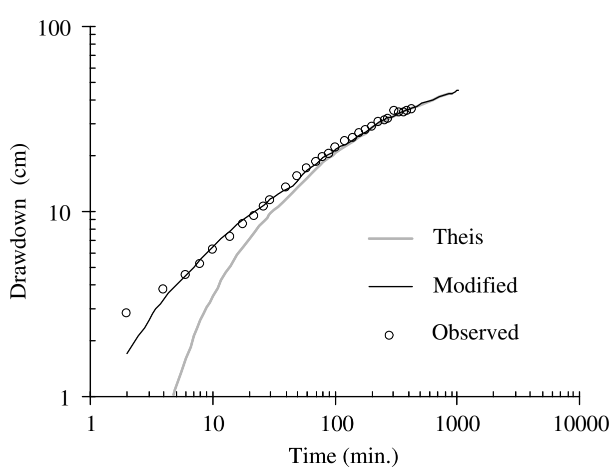

Fig. 7. Change of observed and calculated drawdown at $r = 104$ m (ZK112) with pumping time, in Example 2. The Theis curve is given by Chen and Wu [3].

assumption may not be appropriate when the bending effect of the confining unit is considered. This effect will reduce the total stress transferred to the aquifer and reduce the amount of water that can be expelled from storage.

An analytical model that couples flow in the aquifer and bending of the confining unit was derived using the theory of thin elastic plates. Solutions were given for drawdown of groundwater level and deflection of the overlying formation. It was demonstrated that Theis solution is a special case in which the overlying formation has no rigidity or the aquifer matrix is incompressible. The drawdown estimated from the new equation was compared with the drawdown from the Theis equation. It can be concluded that drawdown from the Theis equation is less than the drawdown affected by bending effect of the confining unit. The difference increases when the radial distance is small and time is short. Both theoretical and field examples were presented to demonstrate the bending effect and its impact on water level response in the pumped aquifer. The field case study illustrated that, when the confining unit is well consolidated or the aquifer matrix is highly compressible, the bending effect is noticeable during hours of pumping and affect drawdown at the distance over 100 m from the pumping well. The study in this paper can provide additional insights in understanding the difference between the observed drawdown and that based on the traditional Theis equation when analyzing early drawdown data in a confined aquifer.

Acknowledgment

The authors are grateful to Hongbin Zhan for his suggestions on the early draft of the paper. We appreciate Andrew Barry, Dong-Sheng Jeng, Todd Rasmussen and an anonymous reviewer whose constructive comments led to a significant improvement of this paper.

Appendix A. Derivation of the solutions (25) and (26)

Application of the Hankel transform method to s and w

$$\bar{s} = \int_0^\infty rsJ_0(\beta r)\,dr, \quad \bar{w} = \int_0^\infty rwJ_0(\beta r)\,dr \tag{A.1}$$

where $J_0(x)$ is the first kind of Bessel function of zero order, (19) and (20) become

$$(1 + c\beta^4)\bar{w} = \mu_m\bar{s} \tag{A.2}$$

$$\frac{Q}{2\pi} - T\beta^2\bar{s} = \frac{\partial\bar{w}}{\partial t} + \mu_w\frac{\partial\bar{s}}{\partial t} \tag{A.3}$$

Substituting (A.2) into (A.3) yields

$$a(1+c\beta^4)\frac{Q}{2\pi T} - a\beta^2(1+c\beta^4)\bar{s} = \left(1+\frac{\mu_w}{S}c\beta^4\right)\frac{\partial\bar{s}}{\partial t} \tag{A.4}$$

where a and b are defined as (28).

The solution to (A.4) satisfying the initial conditions is

$$\bar{s} = \frac{Q}{2\pi T}\int_0^t \frac{\lambda(\beta)}{\beta^2}e^{-\lambda(\beta)(t-\tau)}\,d\tau = \frac{Q}{2\pi T}\frac{1-e^{-\lambda(\beta)t}}{\beta^2} \tag{A.5}$$

in which the function $\lambda(\beta)$ are defined as (27). Substituting (A.5) into (A.2) yields

$$\bar{w} = \frac{Q}{2\pi}\left(1-\frac{\mu_w}{S}\right)\frac{1-e^{-\lambda(\beta)t}}{a\beta^2(1+c\beta^4)} \tag{A.6}$$

To obtain s and w, use inverse Hankel transform as follows

$$s = \int_0^\infty \beta\bar{s}J_0(\beta r)\,d\beta, \quad w = \int_0^\infty \beta\bar{w}J_0(\beta r)\,d\beta \tag{A.7}$$

and (25) and (26) are obtained.

Appendix B. Derivation of (32) and (33)

When $c = 0$, (27) becomes

$$\lambda(\beta) = a\beta^2 \tag{B.1}$$

and (26) becomes

$$s = \frac{Q}{2\pi T}\int_0^\infty \frac{1-e^{-a\beta^2 t}}{\beta}J_0(\beta r)\,d\beta \tag{B.2}$$

Substituting the variables defined in (29) into (B.2) and using the well function defined in (30) give

$$s = \frac{Q}{4\pi T}M\left(\frac{1}{u},0,\frac{\mu_w}{S}\right) = \frac{Q}{4\pi T}\int_0^\infty \frac{1-e^{-x/4u}}{x}J_0(\sqrt{x})\,dx \tag{B.3}$$

Furthermore, substituting (B.1) into (A.5) yields

$$\bar{s} = \frac{Q}{4\pi T}\int_0^t 2ae^{-a\beta^2(t-\tau)}\,d\tau \tag{B.4}$$

Directly applying the inverse Hankel transform to (B.4) gives

$$s = \frac{Q}{4\pi T}\int_0^t 2af(r)\,d\tau \tag{B.5}$$

in which the function $f(r)$ is defined as

$$f(r) = \int_0^\infty e^{-a\beta^2(t-\tau)}\beta J_0(\beta r)\,d\beta \tag{B.6}$$

It is obvious that

$$f(0) = \frac{1}{2a(t-\tau)} \tag{B.7}$$

From the property of Bessel functions we have

$$\frac{df}{dr} + \frac{r}{2a(t-\tau)}f = 0 \tag{B.8}$$

The solution of (B.8) satisfying (B.7) is

$$f(r) = \frac{1}{2a(t-\tau)}e^{-r^2/4a(t-\tau)} \tag{B.9}$$

Then, (B.5) becomes

$$s = \frac{Q}{4\pi T}\int_0^t \frac{1}{t-\tau}e^{-r^2/4a(t-\tau)}\,d\tau \tag{B.10}$$

With the definitions

$$y = \frac{r^2}{4a(t-\tau)}, \quad u = \frac{r^2}{4at} \tag{B.11}$$

(B.10) can be rewritten as

$$s = \frac{Q}{4\pi T}\int_u^\infty \frac{e^{-y}}{y}\,dy = \frac{Q}{4\pi T}W(u) \tag{B.12}$$

Comparison of the Eqs. (B.3) and (B.2) leads to (32).

For the case $c \to \infty$ there is

$$\lambda(\beta) = \frac{T}{\mu_w}\beta^2 \tag{B.13}$$

Replacing a with T/μ_w in (B.1) and (B.2) leads to (33).

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