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Analysis of soil consolidation by vertical drains with double porosity model

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SUMMARY

The soil around a drain well is traditionally divided into smeared zone and undisturbed zone with constant hydraulic conductivity. In reality, hydraulic conductivity of the soil changes continuously and it may not be always appropriate to approximate its distribution with two zones. In this study, the horizontal hydraulic conductivity of the soil is described by an arbitrary function of radial distance. The horizontal flow under equal strain condition is analysed for a soil–drain system with a circular or regular polygonal boundary. It is found that the horizontal flow can be generally characterized with a linear equation in which the flow rate of water through soil–drain interface is proportional to the difference between the average excess pore pressure in the soil and the excess pore pressure in the drain well. The water exchange between the drain and the soil is analogous to that between fractures and matrix in a double porosity system, a popular conceptual model of fracture rocks. On the basis of this characterization, a simplified approach to analyse soil–drain systems is developed with one-dimensional double porosity model (*DPM*). Analytical solutions for both fully and partially penetrating drains are derived. The solution for partially penetrating drains is compared with both numerical and approximate analytical results in literature. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: consolidation; foundation; vertical drain; double porosity

INTRODUCTION

Vertical drains have been used world wide to accelerate the consolidation process of fine-grained soils. In the past decades, various analytical solutions were developed for consolidation of soft foundation by vertical drains [1–8]. Well resistance and smear effect were taken account for in different ways. These solutions have contributed to the design and assessment of vertical drains in soft foundation.

When the smear effect was considered, the soil around a drain well was commonly divided into two annular zones: smeared and undisturbed zones. The boundary between the two zones and permeability of the smeared zone are key parameters and need to be specified before using the solutions, but in practice it is difficult or subjective to specify their values (e.g. References

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[9, 10]). In reality, the soil permeability changes continuously and it may not be appropriate to approximate the permeability distribution with two zones. To remedy the blemish, a more vigorous approach is required.

For partially penetrating drains, Hart *et al.* [11] gave an approximate equation to calculate the overall degree of consolidation. On basis of this equation, Zeng and Xie [6] presented an improved approximate equation. Tang and Onitsuka [7] developed an analytical solution by assuming that the soil layer consisted of two parts. The upper part of the soil layer with the vertical drains was considered as a homogenous ground with both vertical and horizontal flow and the lower part was considered as a single-layered ground with only vertical flow. In their model, the horizontal flow in the lower part of the soil layer, which can be significant near the bottom of drains, was ignored.

In this study, a new model is presented in which the soil–drain system is described as double porosity medium (*DPM*). The model can lead to a more general and simple approach to analyse the behaviours of the soil–drain system than previous solutions. An analytical solution with respect to partially penetrating vertical drains is developed after Tang and Onitsuka's [8] work.

CHARACTERIZATION OF SOIL–DRAIN SYSTEM AS DPM

Similar to the simplification that applied by Barron [1], Hansbo [3] and Xie [4], the model of the soil–drain system in this study is constructed with dominative assumptions as follows:

(1) Equal-strain hypothesis proposed by Barron [1] and later modified by Xie [4] and Tang and Onitsuka [8] is adopted.
(2) Soils around vertical drains are isotropic horizontally and uniform vertically.
(3) Loads are applied to the top surface instantaneously and remain constant during consolidation of the soils.
(4) The horizontal flow inside the drain well is neglected.
(5) No flow passes through the lateral surfaces of the single-drain-controlled area and the bottom of the soil layer. In addition, the excess pore pressure at the top surface is zero.

Description of consolidation in equations

Under equal-strain condition, the partial differential equation describing the consolidation of soils by vertical drains was suggested by Barron [1] in the form as follows:

$$C_{\mathrm{h}}\left(\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}\right)+C_{\mathrm{v}}\frac{\partial^2 u}{\partial z^2}=\frac{\partial \bar{u}}{\partial t} \tag{1}$$

As demonstrated by Xie [4] and Tang and Onitsuka [8], Equation (1) is not appropriate to solve the problem coupling the vertical and horizontal flow in the soil. They suggested that the second item of the left-hand side of (1) should be described with average excess pore pressure, $\bar{u}$. Then (1) is modified as,

$$C_{\mathrm{h}}\left(\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}\right)+C_{\mathrm{v}}\frac{\partial^2 \bar{u}}{\partial z^2}=\frac{\partial \bar{u}}{\partial t} \tag{2}$$

This suggestion is used herein. In radial co-ordinate system, (2) can be rewritten as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{k_{\mathrm{h}}(r)}{\gamma_{\mathrm{w}}} r \frac{\partial u}{\partial r}\right] + \frac{\partial}{\partial z}\left(\frac{k_{\mathrm{v}}}{\gamma_{\mathrm{w}}}\frac{\partial \bar{u}}{\partial z}\right) = m_{\mathrm{s}}\frac{\partial \bar{u}}{\partial t} \tag{3}$$

In an orthogonal co-ordinate system, (3) becomes

$$\frac{\partial}{\partial x}\left[\frac{k_{\mathrm{h}}(x,y)}{\gamma_{\mathrm{w}}}\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{k_{\mathrm{h}}(x,y)}{\gamma_{\mathrm{w}}}\frac{\partial u}{\partial y}\right] + \frac{\partial}{\partial z}\left(\frac{k_{\mathrm{v}}}{\gamma_{\mathrm{w}}}\frac{\partial \bar{u}}{\partial z}\right) = m_{\mathrm{s}}\frac{\partial \bar{u}}{\partial t} \tag{4}$$

Introducing a term

$$q(z,t) = \frac{\partial}{\partial z}\left(\frac{k_{\mathrm{v}}}{\gamma_{\mathrm{w}}}\frac{\partial \bar{u}}{\partial z}\right) - m_{\mathrm{s}}\frac{\partial \bar{u}}{\partial t}$$

(3) and (4) become,

$$\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{k_{\mathrm{h}}(r)}{\gamma_{\mathrm{w}}} r \frac{\partial u}{\partial r}\right] + q(z,t) = 0 \tag{5}$$

$$\frac{\partial}{\partial x}\left[\frac{k_{\mathrm{h}}(x,y)}{\gamma_{\mathrm{w}}}\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{k_{\mathrm{h}}(x,y)}{\gamma_{\mathrm{w}}}\frac{\partial u}{\partial y}\right] + q(z,t) = 0 \tag{6}$$

$q(z,t)$ is independent of the radial distance and behaves as the sink–source term in (5) and (6).

Equations (1)–(6) describe the horizontal seepage in the soil zone of the soil–drain system. To solve them the boundary condition associated with the drain well is needed. Considering the well resistance, the boundary condition is given as follows:

$$2\pi r \frac{k_{\mathrm{h}}(r)}{\gamma_{\mathrm{w}}}\frac{\partial u}{\partial r}\bigg|_{r=r_{\mathrm{w}}} = -A_{\mathrm{w}}\frac{k_{\mathrm{w}}}{\gamma_{\mathrm{w}}}\frac{\partial^2 u_{\mathrm{w}}}{\partial z^2} \tag{7}$$

Equation (7) is also the differential equation which describes the vertical seepage in the drain well. The left-hand side of the equation is the sink–source term.

Horizontal flow in soil–drain system with circular boundary

As commonly assumed by previous researchers, it is first assumed that the influential area of a single drain is a cylinder with effective radius r_{e} (Figure 1(a)). As shown in Figure 2, the horizontal hydraulic conductivity of the soil with respect to the radial distance r can be described by

$$k_{\mathrm{h}}(r) = k_m f(r) \tag{8}$$

where k_m is the hydraulic conductivity at r_{e} and $f(r)$ is a function which depends on the variation patterns of horizontal hydraulic conductivity in the soil.

Substituting (8) into (5) yields

$$\frac{k_m}{\gamma_{\mathrm{w}}}\frac{1}{r}\frac{\partial}{\partial r}\left[f(r) r \frac{\partial u}{\partial r}\right] + q = 0 \tag{9}$$

The boundary conditions are

$$u(r) = u_{\mathrm{w}}, \quad r = r_{\mathrm{w}} \tag{10}$$

$$\frac{\partial u(r)}{\partial r} = 0, \quad r = r_{\mathrm{e}} \tag{11}$$

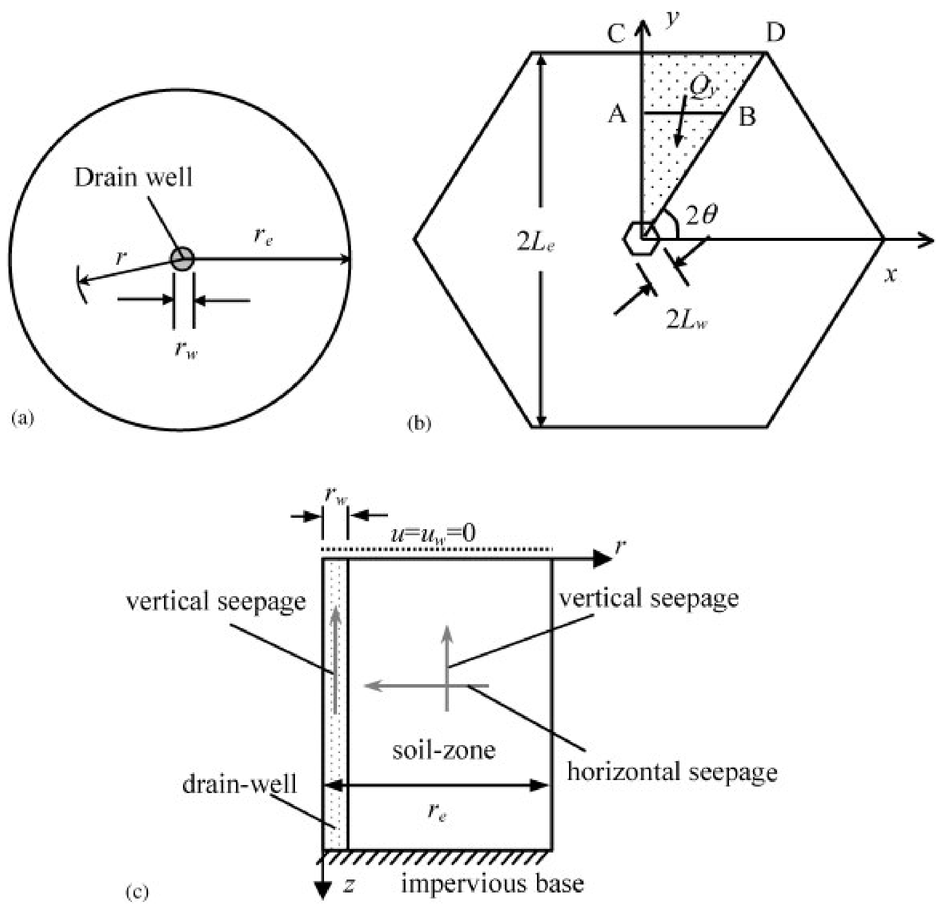

Figure 1. Schematic diagram of a soil–drain system: (a) cross-section with circular boundary; (b) cross-section with polygonal boundary; and (c) vertical profile.

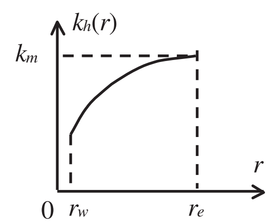

Figure 2. Representation of variation of horizontal hydraulic conductivity in the soil with radial distance for a soil–drain system with circular boundary.

The solution of (9) subject to (10)–(11) is,

$$u(r) = u_{\mathrm{w}} + \frac{\gamma_{\mathrm{w}} q}{2k_m}[r_{\mathrm{e}}^2 A_0(r) - B_0(r)] \tag{12}$$

where

$$A_0(r) = \int_{r_w}^{r} \frac{\mathrm{d}\xi}{\xi f(\xi)}, \quad B_0(r) = \int_{r_w}^{r} \frac{\xi\,\mathrm{d}\xi}{f(\xi)} \tag{13}$$

So the average excess pore pressure between r_w and r_e is

$$\bar{u} = \frac{1}{\pi(r_e^2 - r_w^2)} \int_{r_w}^{r_e} 2\pi r u(r)\,\mathrm{d}r = u_w + \frac{q\gamma_w}{k_m(r_e^2 - r_w^2)}(A_1 r_e^2 - B_1) \tag{14}$$

where

$$A_1 = \int_{r_w}^{r_e} r A_0(r)\,\mathrm{d}r, \quad B_1 = \int_{r_w}^{r_e} r B_0(r)\,\mathrm{d}r \tag{15}$$

The rate of water flow Q_w over a unit thickness entering into the drain well through the soil–drain interface is also the sink–source term of (7) for seepage in the drain well

$$Q_w(z,t) = 2\pi r \frac{k_h(r)}{\gamma_w} \frac{\partial u}{\partial r}\bigg|_{r=r_w} \tag{16}$$

From water balance, or substituting (12) into (16), we have

$$Q_w(z,t) = q\pi(r_e^2 - r_w^2) \tag{17}$$

Combining (14) and (17) yields

$$Q_w(z,t) = C_q[\bar{u}(z,t) - u_w(z,t)] \tag{18}$$

where

$$C_q = \frac{k_m \pi (r_e^2 - r_w^2)^2}{\gamma_w (A_1 r_e^2 - B_1)} \tag{19}$$

Equation (18) shows that the flow rate of water entering into the drain well at depth z and time t is proportional to the difference between the average excess pore pressure in the soil and the excess pore pressure in the drain well at the same depth z and at the time t. We call the parameter C_q as the water-exchange coefficient of the soil–drain system.

In the traditional approach, the distribution of horizontal hydraulic conductivity in smeared zone and undisturbed zone, can be represented by $f(r)$ as follows:

$$f(r) = \begin{cases} 1/\chi, & r_w \leqslant r < r_s \\ 1.0, & r_s < r \leqslant r_e \end{cases} \tag{20}$$

where $\chi = k_m/k_s$. Substituting (20) into (19) we have

$$C_q = \frac{2\pi k_m}{\gamma_w F(n,s,\chi)} \left(1 - \frac{1}{n^2}\right) \tag{21}$$

where

$$F(n,s,\chi) = \frac{n^2}{n^2-1}\left[\ln(n) - \frac{3}{4} + \frac{4n^2-1}{4n^4} + (\chi - 1)\left(\ln s + \frac{1-s^2}{n^2} + \frac{s^4-1}{4n^4}\right)\right] \tag{22}$$

where $n = r_e/r_w$, $s = r_s/r_w$.

Horizontal flow in a soil–drain system with regular polygonal boundary

For the drains arranged in triangular or foursquare pattern, the actual horizontal boundary of the single-drain-controlled area is a regular polygon. As an example, the triangular pattern (Figure 1(b)) is chose for discussion. Due to symmetry of the problem, a representative area (the shaded area in Figure 1(b)) is chosen for discussion. If the number of sides is N, then the representative area is $1/2N$ of the total area.

The interval between two drains is defined as $2L_e$. For convenience of discussion, the cross-section of the drain well is assumed to have the same polygonal shape (Figure 1(b)) as the boundary of the soil–drain system . The radius of inscribed circle of the cross-section of the drain is defined as L_w.

It is assumed that the hydraulic conductivity of the soil is a function of y in the representative area:

$$k_h(y) = k_m f(y) \tag{23}$$

where k_m is the hydraulic conductivity at $y = L_e$. The definition of $f(y)$ is similar to that in Equation (8).

It is obvious that the rate of water flow through A–B as shown in Figure 1(b) is

$$Q_{AB} = \frac{k_m f(y)}{\gamma_w} \int_0^{y tg\theta} \frac{\partial u(x, y)}{\partial y} \, dx \Big|_{y=y_A} \tag{24}$$

Introducing $x = yv$ into (24) yields

$$Q_{AB} = \frac{k_m f(y)}{\gamma_w} y \int_0^{tg\theta} \frac{\partial u(yv, y)}{\partial y} \, dv \Big|_{y=y_A} = \frac{y k_m f(y)}{\gamma_w} \frac{\partial}{\partial y} \left(\int_0^{tg\theta} u(yv, y) \, dv \right) \Big|_{y=y_A} \tag{25}$$

Defining

$$w(y) = \int_0^{tg\theta} u(yv, y) \, dv \tag{26}$$

(25) becomes

$$Q(y) = \frac{y k_m f(y)}{\gamma_w} \frac{\partial w(y)}{\partial y} \tag{27}$$

According to water balance and boundary conditions, the rate of water flow through A–B, Q_y, is equal to the sink–source item, q, multiplied by the area between A–B and C–D (Figure 1(b)). So

$$Q(y) = \tfrac{1}{2} q (L_e^2 - y^2) tg\theta \tag{28}$$

Combining (27) and (28) and using the boundary condition $w(L_w) = u_w tg\theta$, we have

$$w(y) = u_w tg\theta + \frac{1}{2} \frac{q \gamma_w}{k_m} [A_0'(y) L_e^2 - B_0'(y)] tg\theta \tag{29}$$

where

$$A_0'(y) = \int_{L_w}^{y} \frac{d\xi}{\xi f(\xi)}, \quad B_0'(y) = \int_{L_w}^{y} \frac{\xi \, d\xi}{f(\xi)} d\xi \tag{30}$$

In addition, the normal excess pore pressure in the representative area is

$$\bar{u} = \frac{2}{(L_e^2 - L_w^2)tg\theta} \int_{L_w}^{L_e} \int_0^{ytg\theta} u(x,y)\,dx\,dy = \frac{2}{(L_e^2 - L_w^2)tg\theta} \int_{L_w}^{L_e} \left[y \int_0^{tg\theta} u(yv,y)\,dv \right] dy \quad (31)$$

Substituting (26) and (29) into (31) yields

$$\bar{u} = u_w + \frac{\gamma_w q}{k_m(L_e^2 - L_w^2)}[A_1' L_e^2 - B_1'] \quad (32)$$

where

$$A_1' = \int_{L_w}^{L_e} yA_0'(y)\,dy, \quad B_1' = \int_{L_w}^{L_e} yB_0'(y)\,dy \quad (33)$$

Due to symmetry of the problem, the normal excess pore pressure presented in (31) and (32) is also the normal excess pore pressure in the total controlled area as shown in Figure 1(b).

The total rate of water flow Q_w entering into the drain well, as the sink–source term in (7), can be calculated from (28) while $y = L_w$,

$$Q_w(z,t) = Nq(L_e^2 - L_w^2)tg\theta = q(L_e^2 - L_w^2)Ntg(\pi/N) \quad (34)$$

Combining (32) and (34) we have,

$$Q_w(z,t) = C_q'[\bar{u}(z,t) - u_w(z,t)] \quad (35)$$

where

$$C_q' = \frac{k_m(L_e^2 - L_w^2)^2 Ntg(\pi/N)}{\gamma_w(A_1' L_e^2 - B_1')} \quad (36)$$

The meaning of (35) is similar to (18). (36) gives the water-exchange coefficient of the soil–drain system with regular polygonal boundary.

It is obvious that when $N \to \infty$, the regular polygon becomes a circle, (36) will be reduced to (19).

Discussion

The water exchange between the drain and the soil in a soil–drain system is analogous to that between fractures and matrix in a double porosity system, a popular conceptual model in handling the fluid flow in fracture rocks. On the basis of such analog, a simplified approach to analyse soil–drain systems can be developed with one-dimensional DPM.

Equations (18) and (35) suggests that the horizontal flow in the soil zone can be described by a simple linear equation, instead of a differential equation. As a result, the three-dimensional consolidation in the soil–drain system is simplified to a one-dimensional problem presented as DPM. The rate of water flow into the drain well Q_w at a depth is proportional to the difference between the average excess pore pressure in the soil, $\bar{u}$, and the excess pore pressure in the drain well, u_w. The water-exchange coefficient of the soil–drain system, C_q, is the only parameter required in describing this linear relationship. C_q is a lumped parameter dependent on the geometry of soil–drain system and the distribution of horizontal permeability.

ANALYTICAL SOLUTIONS FOR FULLY AND PARTIALLY PENETRATING DRAINS

Unlike the traditional approaches for analysis of vertical and horizontal flow in the soil–drain system, the one-dimensional DPM uses two variables $\bar{u}$, u_w to express the state of flow at depth z and time t. In the simplified model, horizontal flow is represented by the water-exchange coefficient. This approach is convenient to obtain analytical solutions for fully or partially penetrating drains.

Analytical solution for fully penetrating drains

Assume that a constant load is applied to the top surface instantaneously and the soil–drain system is uniform vertically, the one-dimensional DPM of the soil–drain system can be described as follows:

$$A_s \frac{k_v}{\gamma_w} \frac{\partial^2 \bar{u}}{\partial z^2} - C_q(\bar{u} - u_w) = A_s m_s \frac{\partial \bar{u}}{\partial t}, \quad 0 < z < H, \;\; t > 0 \tag{37}$$

$$A_w \frac{k_w}{\gamma_w} \frac{\partial^2 u_w}{\partial z^2} + C_q(\bar{u} - u_w) = 0, \quad 0 < z < H, \;\; t > 0 \tag{38}$$

$$\bar{u}(z,t) = u_0, \quad u_w(z,t) = u_0, \quad t = 0, \;\; 0 < z < H \tag{39}$$

$$\bar{u}(z,t) = 0, \quad u_w(z,t) = 0, \quad t > 0, \;\; z = 0 \tag{40}$$

$$\frac{\partial \bar{u}(z,t)}{\partial z} = 0, \quad \frac{\partial u_w(z,t)}{\partial z} = 0, \quad t > 0, \;\; z = H \tag{41}$$

Among them, (37) and (38) describe the flow of water in the soil and drain well, respectively; (39) is the initial condition; (40) and (41) represent the boundary conditions at the top and the bottom surfaces, respectively. Equation (38) is derived by introducing (18) or (35) to the sink–source term of (7).

Solution of (37) and (38) subject to (39)–(41) can be obtained through variable separating method as follows:

$$\bar{u} = u_0 \sum_{m=0}^{\infty} \frac{2}{M} \sin\left(\frac{Mz}{H}\right) \exp\left[-\left(\beta_m^s + \frac{C_q}{\beta_0} \frac{\beta_m^w}{C_q + \beta_m^w}\right) t\right] \tag{42}$$

$$u_w = u_0 \sum_{m=0}^{\infty} \frac{C_q}{C_q + \beta_m^w} \frac{2}{M} \sin\left(\frac{Mz}{H}\right) \exp\left[-\left(\beta_m^s + \frac{C_q}{\beta_0} \frac{\beta_m^w}{C_q + \beta_m^w}\right) t\right] \tag{43}$$

where

$$M = (2m+1)\pi/2, \quad m = 0, 1, 2, \ldots$$

$$\beta_m^s = k_v M^2/(\gamma_w m_s H^2) = C_v M^2/H^2$$

$$\beta_m^w = k_w A_w M^2/(\gamma_w H^2)$$

$$\beta_0 = A_s m_s$$

where $C_v = k_v/\gamma_w m_s$, is the vertical coefficient of consolidation of the soil.

Substituting (21) into (42) and (43), the solution is reduced to the solution derived by Xie [4].

Analytical solutions for partially penetrating drains

A partially penetrating drain can be approximated by a fully penetrating drain consisting of two segments, with the upper one being a real drain and the lower one being an imaginary drain which has the same radius as the upper one and the same properties as the background soil (Figure 3). It is assumed that the water in the imaginary drain flows vertically. The water-exchange coefficient of the imaginary drain well C_{q2} is different from that of real drain, C_{q1}. C_{q2} can be obtained by setting $\chi = 1$ in Equation (21).

Assume that a constant load is applied on the top surface instantaneously. The behaviour of this one-dimensional DPM, can be described as follows:

$$A_{\rm s}\frac{k_{\rm v1}}{\gamma_{\rm w}}\frac{\partial^2 \bar{u}_1}{\partial z^2} - C_{q1}(\bar{u}_1 - u_{\rm w1}) = A_{\rm s}m_{\rm s1}\frac{\partial \bar{u}_1}{\partial t}, \quad 0<z<H_{\rm w}, \ \ t>0 \tag{44}$$

$$A_{\rm w}\frac{k_{\rm w}}{\gamma_{\rm w}}\frac{\partial^2 u_{\rm w1}}{\partial z^2} + C_{q1}(\bar{u}_1 - u_{\rm w1}) = A_{\rm w}m_{\rm w1}\frac{\partial u_{\rm w1}}{\partial t}, \quad 0<z<H_{\rm w}, \ \ t>0 \tag{45}$$

$$A_{\rm s}\frac{k_{\rm v2}}{\gamma_{\rm w}}\frac{\partial^2 \bar{u}_2}{\partial z^2} - C_{q2}(\bar{u}_2 - u_{\rm w2}) = A_{\rm s}m_{\rm s2}\frac{\partial \bar{u}_2}{\partial t}, \quad H_{\rm w}<z<H, \ \ t>0 \tag{46}$$

$$A_{\rm w}\frac{k_{\rm w2}}{\gamma_{\rm w}}\frac{\partial^2 u_{\rm w2}}{\partial z^2} + C_{q2}(\bar{u}_2 - u_{\rm w2}) = A_{\rm w}m_{\rm w2}\frac{\partial u_{\rm w2}}{\partial t}, \quad H_{\rm w}<z<H, \ \ t>0 \tag{47}$$

$$\bar{u}_i(z,t) = u_0, \quad u_{\rm wi}(z,t) = u_0, \quad i = 1,2, \ \ t = 0, \ \ 0<z<H \tag{48}$$

$$\bar{u}_1(z,t) = 0, \quad u_{\rm w1}(z,t) = 0, \quad z = 0, \ \ t>0 \tag{49}$$

$$\frac{\partial \bar{u}_2(z,t)}{\partial z} = 0, \quad \frac{\partial u_{\rm w2}(z,t)}{\partial z} = 0, \quad z = H, \ \ t>0 \tag{50}$$

$$\bar{u}_1(z,t) = \bar{u}_2(z,t), \quad u_{\rm w1}(z,t) = u_{\rm w2}(z,t), \quad z = H_{\rm w}, \ \ t>0 \tag{51}$$

$$k_{\rm v1}\frac{\partial \bar{u}_1}{\partial z} = k_{\rm v2}\frac{\partial \bar{u}_2}{\partial z}, \quad k_{\rm w1}\frac{\partial u_{\rm w1}}{\partial z} = k_{\rm w2}\frac{\partial u_{\rm w2}}{\partial z}, \quad z = H_{\rm w}, \ \ t>0 \tag{52}$$

Among the above equations, (44) and (45) describe water flow in the upper soil and partially penetrating drain well, respectively; (46) and (47) describe water flow in the lower soil and the imaginary drain well, respectively; (48) is the initial condition; (49) and (50) are the boundary conditions; (51) and (52) are the continuity conditions at the interface of the partially penetrating and imaginary drain wells. In the model, the vertical compressibility of drains is included. Note that for the imaginary drain well, $m_{\rm w2} = m_{\rm s2}$ and $k_{\rm w2} = k_{\rm v2}$.

Tang and Onitsuka [8] have suggested that the consolidation problem of soil by partially penetrating vertical drains can be solved using the approach of double-layered ground with vertical drains, considering the lower part of the ground as a single soil layer. In Tang–Onitsuka's model, the horizontal flow near the bottom of the drain well in the underlying soil is neglected and the vertical compressibility of the vertical drain is ignored. In this study, the horizontal flow in the lower part and the vertical compressibility of the drain are considered.

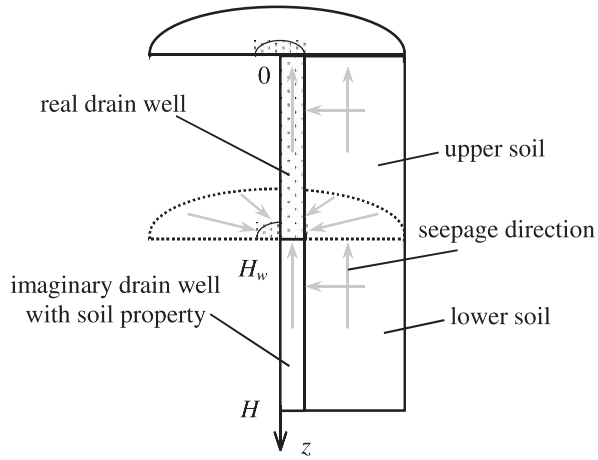

Figure 3. Schematic diagram of DPM model for a soil–drain system with a partially penetrating drain.

A symmetric and integrated description of soil–drain consolidation in mathematical form similar to (44)–(47) will be provided.

Using the same analytical method employed by Tang and Onitsuka's [8], the solution to the problem described by (44)–(52) can be expressed as follows:

$$u_{\mathrm{w}i}(z,t) = u_0 \sum_{m=0}^{\infty} B_m g_{wmi}(z) \mathrm{e}^{-\beta_m t}, \quad i = 1,2 \tag{53}$$

$$\bar{u}_i(z,t) = u_0 \sum_{m=0}^{\infty} B_m g_{mi}(z) \mathrm{e}^{-\beta_m t}, \quad i = 1,2 \tag{54}$$

where

$$g_{\mathrm{w}m1}(z) = a_{m1} \sin\left(\lambda_{m1} \frac{z}{H}\right) + c_{m1} \sinh\left(\xi_{m1} \frac{z}{H}\right) \tag{55}$$

$$g_{m1}(z) = \left(1 - \frac{\beta_m}{C_{\mathrm{w}1}\varphi_{\mathrm{w}1}} + \frac{\lambda_{m1}}{\varphi_{\mathrm{w}1}}\right) a_{m1} \sin\left(\lambda_{m1} \frac{z}{H}\right) + \left(1 - \frac{\beta_m}{C_{\mathrm{w}1}\varphi_{\mathrm{w}1}} - \frac{\xi_{m1}}{\varphi_{\mathrm{w}1}}\right) c_{m1} \sinh\left(\xi_{m1} \frac{z}{H}\right) \tag{56}$$

$$g_{\mathrm{w}m2}(z) = b_{m2} \cos\left[\lambda_{m2}\left(1 - \frac{z}{H}\right)\right] + d_{m2} \cosh\left[\xi_{m2}\left(1 - \frac{z}{H}\right)\right] \tag{57}$$

$$g_{m2}(z) = \left(1 - \frac{\beta_m}{C_{\mathrm{w}2}\varphi_{\mathrm{w}2}} + \frac{\lambda_{m2}^2}{\varphi_2}\right) b_{m2} \cos\left[\lambda_{m2}\left(1 - \frac{z}{H}\right)\right] + \left(1 - \frac{\beta_m}{C_{\mathrm{w}2}\varphi_{\mathrm{w}2}} - \frac{\xi_{m2}^2}{\varphi_{\mathrm{w}2}}\right) d_{m2} \cosh\left[\xi_{m2}\left(1 - \frac{z}{H}\right)\right] \tag{58}$$

where

$$\lambda_{mi} = H\sqrt{\frac{\Xi_{mi} + \sqrt{\Xi_{mi}^2 - 4\beta_m\Theta_{mi}}}{2C_{\mathrm{vi}}}}, \qquad \xi_{mi} = H\sqrt{\frac{-\Xi_{mi} + \sqrt{\Xi_{mi}^2 - 4\beta_m\Theta_{mi}}}{2C_{\mathrm{vi}}}}$$

$$\Xi_{mi} = \beta_m(1+\phi_i) - C_{\mathrm{vi}}(\varphi_{\mathrm{wi}} + \varphi_i), \quad \Theta_{mi} = \frac{\beta_m}{C_{\mathrm{wi}}} - \varphi_{\mathrm{wi}} - \varphi_i\phi_i$$

$$\varphi_{\mathrm{wi}} = \frac{\gamma_{\mathrm{w}}C_{qi}}{A_{\mathrm{w}}k_{\mathrm{wi}}}, \quad \varphi_i = \frac{\gamma_{\mathrm{w}}C_{qi}}{A_{\mathrm{s}}k_{\mathrm{vi}}}, \quad C_{\mathrm{wi}} = \frac{k_{\mathrm{wi}}}{\gamma_{\mathrm{w}}m_{\mathrm{wi}}}, \quad C_{\mathrm{vi}} = \frac{k_{\mathrm{vi}}}{\gamma_{\mathrm{w}}m_{\mathrm{si}}}, \quad \phi_i = C_{\mathrm{vi}}/C_{\mathrm{wi}}$$

In determining the values of a_{m1}, c_{m1}, b_{m2}, d_{m2}, β_m, B_m, a matrix-form equation is introduced to satisfy the continuity conditions shown in (51) and (52):

$$\mathbf{S}_{4\times4}\mathbf{X}^{\mathrm{T}} = 0 \tag{59}$$

where

$$\mathbf{s}_{11} = \sin(\rho\lambda_{m1}), \quad \mathbf{s}_{12} = \sinh(\rho\xi_{m1})$$

$$\mathbf{s}_{13} = -\cos[(1-\rho)\lambda_{m2}], \quad \mathbf{s}_{14} = -\cosh[(1-\rho)\xi_{m2}]$$

$$\mathbf{s}_{21} = \left(1 - \frac{\beta_m}{C_{\mathrm{w1}}\varphi_{\mathrm{w1}}} + \frac{\lambda_{m1}^2}{\varphi_{\mathrm{w1}}}\right)\sin(\rho\lambda_{m1}), \quad \mathbf{s}_{22} = \left(1 - \frac{\beta_m}{C_{\mathrm{w1}}\varphi_{\mathrm{w1}}} - \frac{\xi_{m1}^2}{\varphi_{\mathrm{w1}}}\right)\sinh(\rho\xi_{m1})$$

$$\mathbf{s}_{23} = -\left(1 - \frac{\beta_m}{C_{\mathrm{w2}}\varphi_{\mathrm{w2}}} + \frac{\lambda_{m2}^2}{\varphi_{\mathrm{w2}}}\right)\cos[(1-\rho)\lambda_{m2}], \quad \mathbf{s}_{24} = -\left(1 - \frac{\beta_m}{C_{\mathrm{w2}}\varphi_{\mathrm{w2}}} - \frac{\xi_{m2}^2}{\varphi_{\mathrm{w2}}}\right)\cosh[(1-\rho)\xi_{m2}]$$

$$\mathbf{s}_{31} = k_{\mathrm{w1}}\lambda_{m1}\cos(\rho\lambda_{m1}), \quad \mathbf{s}_{32} = k_{\mathrm{w1}}\xi_{m1}\cosh(\rho\xi_{m1})$$

$$\mathbf{s}_{33} = -k_{\mathrm{w2}}\lambda_{m2}\sin[(1-\rho)\lambda_{m2}], \quad \mathbf{s}_{34} = k_{\mathrm{w2}}\xi_{m2}\sinh[(1-\rho)\xi_{m2}]$$

$$\mathbf{s}_{41} = k_{\mathrm{v1}}\left(1 - \frac{\beta_m}{C_{\mathrm{w1}}\varphi_{\mathrm{w1}}} + \frac{\lambda_{m1}^2}{\varphi_{\mathrm{w1}}}\right)\lambda_{m1}\cos(\rho\lambda_{m1}), \quad \mathbf{s}_{42} = k_{\mathrm{v1}}\left(1 - \frac{\beta_m}{C_{\mathrm{w1}}\varphi_{\mathrm{w1}}} - \frac{\xi_{m1}^2}{\varphi_{\mathrm{w1}}}\right)\xi_{m1}\cosh(\rho\xi_{m1})$$

$$\mathbf{s}_{43} = -k_{\mathrm{v2}}\left(1 - \frac{\beta_m}{C_{\mathrm{w2}}\varphi_{\mathrm{w2}}} + \frac{\lambda_{m2}^2}{\varphi_{\mathrm{w2}}}\right)\lambda_{m2}\sin[(1-\rho)\lambda_{m2}]$$

$$\mathbf{s}_{44} = k_{\mathrm{v2}}\left(1 - \frac{\beta_m}{C_{\mathrm{w2}}\varphi_{\mathrm{w2}}} - \frac{\xi_{m2}^2}{\varphi_{\mathrm{w2}}}\right)\xi_{m2}\sinh[(1-\rho)\xi_{m2}]$$

$$\rho = H_{\mathrm{w}}/H, \qquad \mathbf{X} = [a_{m1} \quad c_{m1} \quad b_{m2} \quad d_{m2}]$$

In order to obtain non-zero solutions of vector **X**, the following equation must be satisfied:

$$\det \mathbf{S}_{4\times4} = 0 \tag{60}$$

Then the values of β_m are obtained in which m is a serial number ($m = 1, 2, 3, \ldots, \infty$). Substituting each value of β_m into (59) and letting $a_{m1} = 1$ the corresponding values of c_{m1}, b_{m2}, d_{m2} can be obtained.

Similar to Tang and Onitsuka's [8] solution, B_m is derived from the orthogonal relation of the system as follows:

$$B_m = \frac{m_{s1} \int_0^{H_w} g_{m1}(z)\,dz + m_{s2} \int_{H_w}^{H} g_{m2}(z)\,dz}{m_{s1} \int_0^{H_w} g_{m1}^2(z)\,dz + m_{s2} \int_{H_w}^{H} g_{m2}^2(z)\,dz} = \frac{m_{s1} W_{m1} + m_{s2} W_{m2}}{m_{s1} W_{m3} + m_{s2} W_{m4}} \tag{61}$$

where

$$\begin{aligned} W_{m1} &= \frac{1}{H} \int_0^{H_w} g_{m1}(z)\,dz \\ &= \frac{a_{m1}}{\lambda_{m1}} \left(1 - \frac{\beta_m}{C_{w1}\varphi_{w1}} + \frac{\lambda_{m1}^2}{\varphi_{w1}}\right)[1 - \cos(\rho\lambda_{m1})] \\ &\quad + \frac{c_{m1}}{\xi_{m1}} \left(1 - \frac{\beta_m}{C_{w1}\varphi_{w1}} - \frac{\xi_{m1}^2}{\varphi_{w1}}\right)[\cosh(\rho\xi_{m1}) - 1] \end{aligned}$$

$$\begin{aligned} W_{m2} &= \frac{1}{H} \int_{H_w}^{H} g_{m2}(z)\,dz \\ &= \frac{b_{m2}}{\lambda_{m2}} \left(1 - \frac{\beta_m}{C_{w2}\varphi_{w2}} + \frac{\lambda_{m2}^2}{\varphi_{w2}}\right)\sin[(1-\rho)\lambda_{m2}] \\ &\quad + \frac{d_{m2}}{\xi_{m2}} \left(1 - \frac{\beta_m}{C_{w2}\varphi_{w2}} - \frac{\xi_{m2}^2}{\varphi_{w2}}\right)\sinh[(1-\rho)\xi_{m2}] \end{aligned}$$

$$W_{m3} = W_{m3}^{\mathrm{I}} + W_{m3}^{\mathrm{II}} + W_{m3}^{\mathrm{III}}$$

$$W_{m3}^{\mathrm{I}} = \frac{1}{2}\left(1 - \frac{\beta_m}{C_{w1}\varphi_{w1}} + \frac{\lambda_{m1}^2}{\varphi_{w1}}\right)^2 a_{m1}^2 \left[\rho - \frac{1}{2\lambda_{m1}}\sin(2\rho\lambda_{m1})\right]$$

$$W_{m3}^{\mathrm{II}} = \frac{1}{2}\left(1 - \frac{\beta_m}{C_{w1}\varphi_{w1}} - \frac{\xi_{m1}^2}{\varphi_{w1}}\right)^2 c_{m1}^2 \left[\frac{1}{2\xi_{m1}}\sinh(2\rho\xi_{m1}) - \rho\right]$$

$$\begin{aligned} W_{m3}^{\mathrm{III}} &= 2\left(1 - \frac{\beta_m}{C_{w1}\varphi_{w1}} + \frac{\lambda_{m1}^2}{\varphi_{w1}}\right)\left(1 - \frac{\beta_m}{C_{w1}\varphi_{w1}} - \frac{\xi_{m1}^2}{\varphi_{w1}}\right)\frac{a_{m1}c_{m1}}{\lambda_{m1}^2 + \xi_{m1}^2} \\ &\quad [\xi_{m1}\sin(\rho\lambda_{m1})\cosh(\rho\xi_{m1}) - \lambda_{m1}\cos(\rho\lambda_{m1})\sinh(\rho\xi_{m1})] \end{aligned}$$

$$W_{m4} = W_{m4}^{\mathrm{I}} + W_{m4}^{\mathrm{II}} + W_{m4}^{\mathrm{III}}$$

$$W_{m4}^{\mathrm{I}} = \frac{1}{2}\left(1 - \frac{\beta_m}{C_{w2}\varphi_{w2}} + \frac{\lambda_{m2}^2}{\varphi_{w2}}\right)^2 b_{m2}^2 \left\{1 - \rho + \frac{1}{2\lambda_{m2}}\sin[2(1-\rho)\lambda_{m2}]\right\}$$

$$W_{m4}^{\mathrm{II}} = \frac{1}{2}\left(1 - \frac{\beta_m}{C_{w2}\varphi_{w2}} - \frac{\xi_{m2}^2}{\varphi_{w2}}\right)^2 d_{m2}^2 \left[1 - \rho + \frac{1}{2\xi_{m2}}\sinh[2(1-\rho)\xi_{m2}]\right]$$

$$W_{m4}^{\mathrm{III}} = 2\left(1 - \frac{\beta_m}{C_{w2}\varphi_{w2}} + \frac{\lambda_{m2}^2}{\varphi_{w2}}\right)\left(1 - \frac{\beta_m}{C_{w2}\varphi_{w2}} - \frac{\xi_{m2}^2}{\varphi_{w2}}\right)\frac{b_{m2}d_{m2}}{\lambda_{m2}^2 + \xi_{m2}^2}$$

$$\{\xi_{m2}\cos[(1-\rho)\lambda_{m2}]\sinh[(1-\rho)\xi_{m2}] - \lambda_{m2}\sin[(1-\rho)\lambda_{m2}]\cosh[(1-\rho)\xi_{m2}]\}$$

Finally, the average degrees of soil consolidation for the upper and lower parts of the soil–drain system are,

$$\bar{U}_1(t) = 1 - \frac{1}{H_w}\int_0^{H_w}\frac{\bar{u}_1(z,t)}{u_0}\,dz = 1 - \frac{1}{\rho u_0}\sum_{m=0}^{\infty} B_m W_{m1} e^{-\beta_m t} \tag{62}$$

$$\bar{U}_2(t) = 1 - \frac{1}{H - H_w}\int_{H_w}^{H}\frac{A_s\bar{u}_2(z,t) + A_w u_{w2}}{u_0(A_w + A_s)}\,dz$$

$$= 1 - \frac{1}{u_0(A_w + A_s)(1-\rho)}\sum_{m=0}^{\infty} B_m(A_s W_{m2} + A_w V_m)e^{-\beta_m t} \tag{63}$$

where

$$V_m = \frac{1}{H}\int_{H_w}^{H} g_{wm2}(z)\,dz = \frac{b_{m2}}{\lambda_{m2}}\sin[(1-\rho)\lambda_{m2}] + \frac{d_{m2}}{\xi_{m2}}\sinh[(1-\rho)\xi_{m2}] \tag{64}$$

The overall degree of soil consolidation defined by pore pressure is

$$\bar{U}(t) = \rho\bar{U}_1(t) + (1-\rho)\bar{U}_2(t) \tag{65}$$

COMPARISON OF THE ANALYTICAL SOLUTION WITH PREVIOUS SOLUTIONS

The same problem has been discussed by Hart *et al.* [11] and Zeng and Xie [6] using approximate equations and by Runesson *et al.* [12] using a numerical method. Runesson *et al.* used a finite element model to calculate the average degree of consolidation for partially penetrating drains with well resistance.

One of the examples used by Runesson *et al.* [12] is investigated again using the solution derived in this paper. In the example, the drain length is half of the total thickness of the soil layer and the well resistance is considered. The parameter values are shown in Table I. Figure 4 shows the variation of the degree of consolidation with normalized time obtained from the analytical solution in this study and the numerical solution by Runesson *et al.* [12] for different normalized depth. As can bee seen, the analytical results match well with the numerical ones.

Table I. Parameter values used in the example.

r_e (m)	r_w (m)	χ	k_w (m/d)	k_v (m/d)	k_m (m/d)	C_w (m^2/d)	$C_v = C_h$ (m^2/d)	ρ
1	0.05	1	1	10^{-4}	10^{-4}	100	0.01	0.5

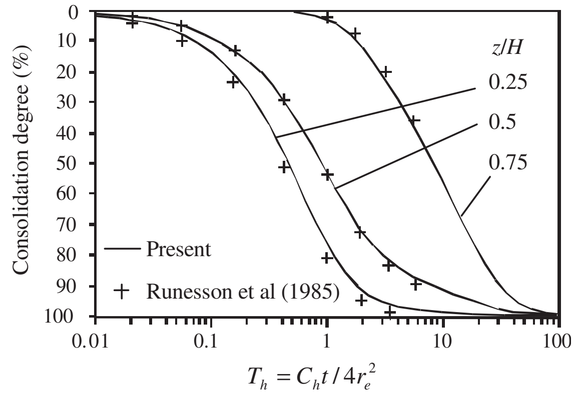

Figure 4. Comparison of consolidation degrees obtained from analytical and numerical solutions, with normalized time at different normalized depth, z/H, for partially penetrating vertical drains.

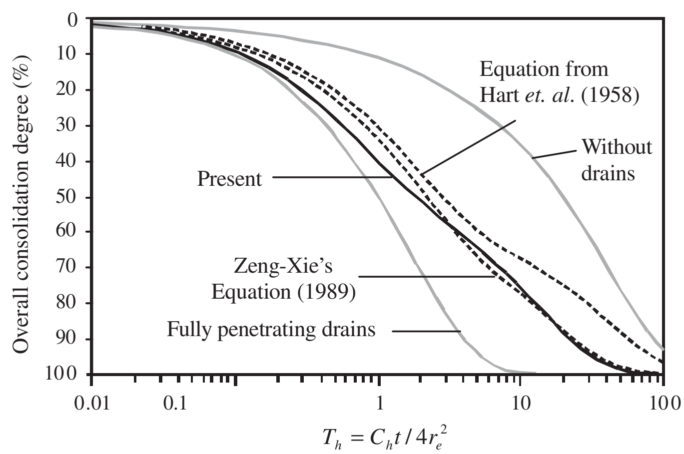

Figure 5. Comparisons of overall consolidation degree obtained from analytical solution and approximate equations along normalized time for partially penetrating vertical drains.

The same example is also evaluated by the approximate solutions presented by Hart *et al.* [11] and Zeng and Xie [6]. The overall degrees of consolidation of soil with normalized time for three different cases (with a partially penetrating drain, with a fully penetrating drain, and without drains) are shown in Figure 5. For partially penetrating drains, the equation presented by Hart *et al.* [11] and Zeng–Xie's equation [6] are not always correct. Zeng–Xie's equation underestimates the consolidation degree in the early period (before the consolidation degree reaches to about 60%) and overestimates in the later time. The equation presented by Hart *et al.* [11] always underestimates the overall consolidation rate. Overall, Zeng–Xie's equation is closer to the analytical solution than that of Hart *et al.*, especially when they are used to predict the time required for the degree of consolidation to achieve 90%.

CONCLUSION

In the study of the consolidation problem of a soil–drain system, previous researchers often assumed that the soil around vertical drains can be divided into two zones: the smeared and the undisturbed zone. In this study, the horizontal hydraulic conductivity of the soil is described by an arbitrary function of radial distance. Under equal strain condition, the horizontal flow in clayey soil around vertical drains can be generally characterized using a simple equation. The equation shows a linear relation between the flow rate of water through the soil–drain interface and the difference of average excess pore pressure in soil and excess pore pressure in the drain well. This characteristic is similar to the flow in double porosity medium such as fractured rock. The water-exchange coefficient can be applied when horizontal hydraulic conductivity of the soils arbitrarily changes with radial distance.

One-dimensional DPM of the soil–drain system is developed and analysed. The analytical solution of fully penetrating drains with well resistance is derived. An integrated description of consolidation by partially penetrating vertical drains is presented and an analytical solution is derived. Horizontal flow in the underlying soil near the drain bottom and the compressibility of the drain-well are considered.

The analytical solution developed with DPM in this study for partially penetrating drains is compared with numerical simulation and approximated equations by previous researchers. The results from the analytical solution derived in this study match well with those from numerical simulation by Runesson *et al.* [12]. Compared to the analytical solution given by DPM, the previous approximate equations are not always accurate.

NOTATION

A_s	plan area of soil in soil–drain system
A_w	plan area of drain well
C_h	average horizontal consolidation coefficient of soil
C_q	water-exchange coefficient of soil–drain system
C_v	average vertical consolidation coefficient of soil
C_w	vertical consolidation coefficient of drain well
$f(r), f(y)$	functions describing variation of k_h in relative
H, H_w	depth of ground and length of vertical drains, respectively
k_h	horizontal hydraulic conductivity of soil
k_m	value of k_h at outer boundary of soil–drain system
k_s	value of k_h in smeared zone
k_v	average vertical hydraulic conductivity of soil
k_w	hydraulic conductivity of drain well
L_e	radius of inscribed circle of soil–drain system with regular polygonal boundary
L_w	radius of inscribed circle of regular polygonal-shaped drain well
m_s	average coefficient of vertical compressibility of soil
m_w	coefficient of vertical compressibility of drain well
n	$n = r_e/r_w$
N	number of sides of regular polygon

$q(z,t)$	source–sink term of horizontal flow in soil
$Q_w(z,t)$	flow rate of water through soil–drain interface along unite depth
r	radial distance to centre of a drain well
r_e	radius of soil–drain system with circular boundary
r_s	radius of interface of smeared zone and undisturbed zone
r_w	radius of drain well
s	$s = r_s/r_w$
t	time
T_h	time factor of consolidation
u	excess pore pressure in soil
$\bar{u}$	average excess pore pressure in soil at a depth
u_0	initial excess pore pressure
u_w	excess pore pressure in drain well
$\bar{U}$	overall consolidation degree defined by pore pressure
$\bar{U}_1$, $\bar{U}_2$	average consolidation degrees of upper and lower parts of soil layer with partially penetrating drains, respectively
x,y	dimensional co-ordinates in horizontal
z	depth under ground surface
γ_w	specific weight of water
χ	$\chi = k_m/k_s$
θ	$\theta = \pi/N$, in regular polygon
ρ	$\rho = H_w/H$

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