Data-Analyses Methods for Determining Two-Dimensional Dispersive Parameters

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Abstract

Two methods for calculating two-dimensional hydrodynamic dispersion parameters by analyzing experimental data, the dispersive-plume-area (DPA) method and the linear-graphic (LG) method, are proposed in this paper. The DPA method determines the dispersive parameters by analyzing the dispersive plume area after injection of a salt-water slug into an aquifer. The transverse and longitudinal dispersivity and even the porosity of an aquifer may be derived. The LG method transforms the concentration-time curve into a straight line. Using the slope of the line, dispersivity and velocity can be obtained. Both methods are examined using theoretical considerations and practical examples.

1. Introduction

Two methods of determining two-dimensional dispersive parameters are described in this paper: the dispersive-plume-area (DPA) method and the linear-graphic (LG) method. The DPA method uses concentration contours of a dispersive plume to estimate dispersive parameters. The concentration contours can be obtained from a multiple well observation system or through surface geophysical methods. If the plume is traceable by surface measurements, only a tracer injection well is needed, which could be more economical than the traditional multiwell monitoring network. This method is demonstrated using the data from a tracer test at a landfill near Borden, Ontario, Canada. The LG method requires concentration-time data from two wells, one along the main axis of flow and one off the main axis of plume migration. Using the values for maximum concentration and time, the method transforms the concentration-time curve into a straight line. From the slope of the line, dispersivity and velocity can be obtained. This method is demonstrated using tracer test data from a laboratory experiment.

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2. Basic Equations

For this development, the aquifer is assumed to be homogenous and isotropic, and ground-water flow is one-dimensional. If the direction of flow is taken as parallel to the x-axis and a tracer injection point (well) is located at the origin, then the movement of the tracer plume after instantaneous injection into the aquifer can be described in rectangular coordinates by (Fried, 1975):

$$C(x, y, t) = \frac{m/\mu}{4\pi t(D_l D_t)^{1/2}} \exp\left[-\frac{(x - ut)^2}{4D_l t} - \frac{y^2}{4D_t t}\right] \quad (1)$$

where C is the concentration of the tracer point (x, y) and time t (ML^{-3}); D_l, D_t are the longitudinal and transverse dispersive coefficients, respectively (L^2/T); μ is porosity (dimensionless); m is the mass of the tracer injected instantaneously over unit thickness of the aquifer (M/L); u is the flow velocity (L/T).

The dispersive coefficients and dispersivities are assumed to conform to the relationship:

$$D_l = \alpha_l u \qquad D_t = \alpha_t u \quad (2)$$

where α_l, α_t are the longitudinal and transverse dispersivities, respectively (L).

3. Dispersion-Plume-Area Method

There have been several methods proposed to obtain two-dimensional hydrodynamic dispersion parameters from field experimental data (Fried, 1975), but each of these methods requires that the data be from a multiple well monitoring network in which one well is used for tracer injection and the other wells for collection of water samples.

The disadvantages are that observation wells are expensive, and the process of obtaining concentration data is complicated. Also, monitoring wells should be located in a distribution downgradient of the injection well. Precise determination of the direction of the flow is difficult. Once a monitoring array is established and a tracer is injected into a well, if the estimated flow direction is incorrect, the tracer may move in a direction away from the observation wells.

The data-analysis method proposed in this section is called the dispersive-plume-area (DPA) method. This method does not require direct concentration data at given points, but only the overall movement of the dispersive plume of the tracer. The movement of the plume could be followed by surface electrical resistivity measurements which are cheaper than multiple well observation systems and can be easily adjusted for changes in direction of tracer movement. Surface resistivity measurements have been well-established by previous researchers. Fried (1975) used field examples to describe single well geophysical methods to determine ground-water velocities. MacFarlane et al. (1983) used an electrical conductance method to delineate the plume near a landfill in Borden, Ontario, Canada. White (1988) used surface resistivity measurements to determine the direction and velocity of ground-water flow and the hydraulic conductivity of an aquifer.

The DPA method is illustrated by using the data from a tracer test completed at a landfill near Borden, Ontario, Canada. The tracer test was made by Sudicky et al. (1983) who obtained the dispersive parameters using traditional analytical methods. The agreement between the parameters acquired by the DPA method and those by Sudicky et al. (1983) shows the new method is applicable.

3.1 Theoretical Development

Rearranging equation (1) yields:

$$\frac{(x-ut)^2}{4D_l t} + \frac{y^2}{4D_t t} = \ln \frac{m/\mu}{4\pi t C (D_l D_t)^{1/2}} \tag{3}$$

This equation shows that the distribution of tracer in an aquifer takes the shape of an ellipse with its center at the point (ut, 0). The ellipse is moving forward with the speed u. Rearranging equation (3) expressions for lengths of the major and minor axes of the ellipse can be obtained:

$$a = [4D_l t \ln \frac{m/\mu}{4\pi t C (D_l D_t)^{1/2}}]^{1/2} \tag{4}$$

$$b = [4D_t t \ln \frac{m/\mu}{4\pi t C (D_l D_t)^{1/2}}]^{1/2} \tag{5}$$

where a, b are half lengths of the major and minor axes of the ellipse, respectively.

From equations (4) and (5), it is easy to determine:

$$a/b = (D_l/D_t)^{1/2} \tag{6}$$

Equation (6) indicates that the ratio of the lengths of the two axes is a constant and reflects the difference in magnitude between the longitudinal and transverse dispersion coefficients.

The area of the ellipse enclosed by the contour with concentration C at time t can be described as:

$$S = \pi ab = 4\pi t (D_l D_t)^{1/2} \ln \frac{m/\mu}{4\pi t C (D_l D_t)^{1/2}} \tag{7}$$

Equation (7) can be rearranged as:

$$S = 4\pi t (D_l D_t)^{1/2} \ln \frac{m/\mu}{4\pi t (D_l D_t)^{1/2}} - 4\pi t (D_l D_t)^{1/2} \ln C \quad \ldots (8)$$

Equation (8) indicates that at time t the relationship between the area S and the concentration C is linear in a semilogarithmic coordinate system. The slope of the line is:

$$k = -4\pi t (D_l D_t)^{1/2} \tag{9}$$

Equation (9) can be rewritten as:

$$-k/4\pi t = (D_l D_t)^{1/2} \tag{10}$$

Solving equations (6) and (10), D_l, D_t can be obtained. If one also measures the $S = 0$ intercept (let it be $d = \ln C$) of the line, then equation (8) becomes:

$$-k \ln \frac{m/\mu}{-k} + kd = 0 \tag{11}$$

From equation (11), an expression for porosity can be obtained:

$$\mu = -m/(ke^d) \tag{12}$$

If the value of porosity is known, equation (12) can be used to estimate the mass injected into the aquifer.

In addition, equation (7) can also be rewritten as:

$$S/t = 4\pi (D_l D_t)^{1/2} \ln \frac{m/\mu}{4\pi C (D_l D_t)^{1/2}} - 4\pi (D_l D_t)^{1/2} \ln t \quad \ldots (13)$$

This shows that if the areas enclosed by contours of a given concentration can be obtained, the relationship between S/t and t is linear in a semilogarithmic coordinate system. A method similar to that described above can be used to estimate the dispersive coefficients after measuring the slope and intercept of the line.

3.2 Field Example

Near Borden, Ontario, Canada, a tracer test was conducted to investigate the migration of contaminants in ground water near a landfill by Sudicky et al. (1983). Approximately 0.7 m^3 of salt water with a chloride concentration of 580.7 mg/l was injected into the ground water. After injection, the tracer slug gradually split into two plumes moving forward with different velocities. One of them, with an average ground-water flow rate of 2.9×10^{-6} m/s, evolved into a Gaussian form. The spatial distribution of chloride after 121 days of transport is shown in Figure 1. The injection and observation well system, shown in Figure 1, consisted of nearly 70 monitoring points. After all the point concentration data was collected, Sudicky et al. (1983)

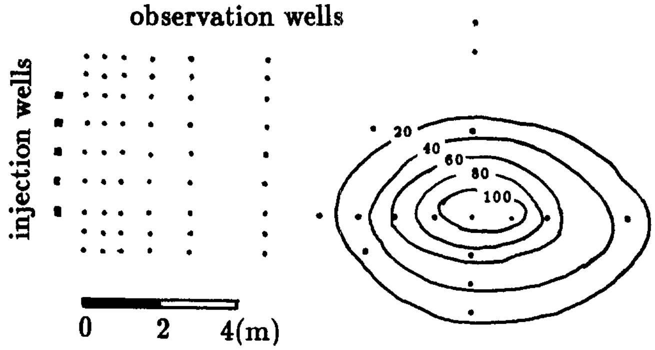

Fig. 1. Chloride distribution in ground water at the Borden Site 121 days after tracer injection [after Sudicky et al. (1983)].

obtained aquifer dispersivities by trial-and-error matching of observed data using an analytical solution:

$$\alpha_l = 0.08 \text{ m}, \qquad \alpha_t = 0.03 \text{ m}$$

The same problem can be solved by the DPA method. First, the lengths of the major and minor axes of the ellipse corresponding to different contours with different concentrations can be measured from Figure 1. After converting the scaled measurements to actual dimensions, the areas and ratios between the major and minor axes of the plume can be calculated. All these data are enumerated in Table 1.

A semilogarithmic plot of the calculated plume area versus concentration produced the relationship shown in Figure 2. The slope of the S-ln C line is measured as $k = -18.45$. Using equation (10) with $t = 121$ days yields:

$$D_l D_t = 3.102530 \tag{14}$$

From Table 1, an average value for a/b can be obtained. Then by using equation (6), one can have:

$$D_l / D_t = 0.000147 \tag{15}$$

Coupling equations (14) and (15), D_l and D_t can be calculated as:

$$D_l = 0.2138\ (m^2/d) \qquad D_t = 0.0069\ (m^2/d)$$

From field measurements the average velocity is known to be:

$$u = 2.9 \times 10^{-6}\ (m/s) = 0.2506\ (m/d)$$

Then by using equations (2), one can have:

longitudinal dispersivity:

$$\alpha_l = D_l/u = 0.09\ (m)$$

transverse dispersivity:

$$\alpha_t = D_t/u = 0.03\ (m)$$

These results are very similar to those obtained by Sudicky et al., but the method is much simpler and more easily understood than the method used by Sudicky et al. (1983).

Table 1. Data for the DPA Method

a/b	*C(mg/l)*	*a(m)*	*b(m)*	*lnC*	*S(m²)*
2.239	100	1.119	0.500	4.605	1.758
1.732	80	1.642	0.948	4.382	4.890
1.667	60	2.239	1.343	4.094	9.447
1.604	40	3.172	1.978	3.689	19.711
1.565	20	4.040	2.575	2.996	32.601

The concentration contours were obtained by analyzing water samples from the observation wells because the test monitoring network was designed for conventional tracer tests and other purposes of researching the aquifer. If the contours were obtained by surface resistivity measurements, then the whole process of data collection and analysis could be much more economical and simple.

4. Linear-Graphic Method

The method outlined in this section is called the linear-graphic (LG) method. It is also based on the use of equation (1) and is designed to estimate dispersivity and velocity. Wang et al. (1987) also developed a line graphical method to estimate dispersive parameters. After rearranging equation (1), Wang et al. found that there is a linear relationship between $H'[= d(\ln(Ct))/dt]$ and $T\ (= 1/t^2)$. The disadvantage of this method is that it is very difficult to perform the derivative computation H′ from actual data, which is usually very erratic and noisy. The LG method discussed in this paper uses the values of maximum C_m and corresponding time t_m and then, using much simpler procedures, transforms the observed concentration-time curve into a straight line.

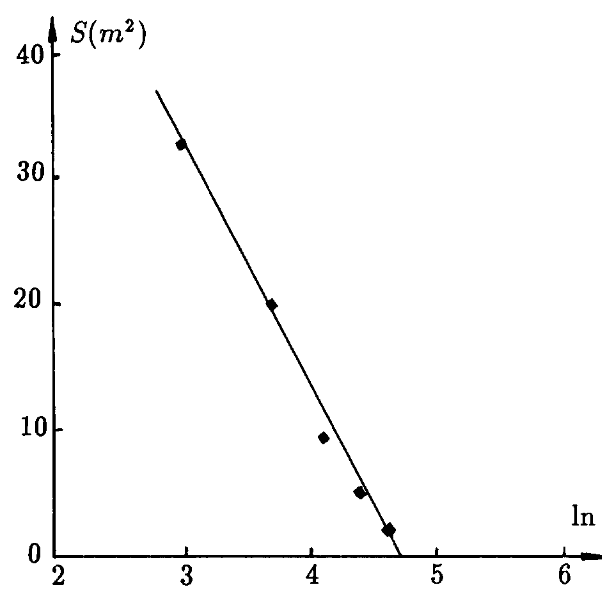

Fig. 2. Plot of S verse lnC developed from Sudicky et al. (1983) tracer dispersion study at the Borden site.

4.1 Theory Development

The theoretical development of the LG method requires taking the derivative of equation (1) with respect to t

$$C' = \frac{m/\mu}{4\pi t^2(D_l D_t)^{1/2}} [\frac{x^2 - u^2t^2}{4D_l t} + \frac{y^2}{4D_t t} - 1] \exp[-\frac{(x-ut)^2}{4D_l t} - \frac{y^2}{4D_t t}] \quad (16)$$

The local maximum values for (1) are found by setting $C' = 0$:

$$\frac{x^2 - u^2t^2}{4D_l t} + \frac{y^2}{4D_t t} - 1 = 0 \quad (17)$$

or

$$(x^2 - u^2t^2)D_t + y^2 D_l - 4D_l D_t t = 0 \quad (18)$$

If the time corresponding to $C' = 0$ is designated as t_m, then equation (18) becomes

$$t_m = \frac{D_l}{u^2} [(4 + \frac{u^2x^2}{D_l^2} + \frac{u^2y^2}{D_l D_t})^{1/2} - 2] \quad (19)$$

The concentration corresponding to t_m is noted as C_m. From equation (1), C_m can be solved as:

$$C_m = \frac{m/\mu}{4\pi t_m(D_l D_t)^{1/2}} \exp[-\frac{(x - ut_m)^2}{4D_l t_m} - \frac{y^2}{4D_t t_m}] \quad \ldots. (20)$$

If the observation point is located on the x-axis, then equation (18) is simplified as:

$$x^2 - u^2t_m^2 - 4D_l t_m = 0 \quad (21)$$

or

$$D_l = (x^2 - u^2t_m^2)/4t_m \quad (22)$$

which can be rewritten as

$$D_l = \frac{x^2}{4t_m[u^2t_m^2/(x^2 - u^2t_m^2) + 1]} \quad (23)$$

letting

$$k = u^2t_m/(x^2 - u^2t_m^2) \quad (24)$$

equation (23) then becomes

$$D_l = \frac{x^2}{4t_m(kt_m + 1)} \quad (25)$$

Substituting (24) into (22) and solving for u yields:

$$u = 2(D_l k)^{1/2} \quad (26)$$

Equations (25) and (26) indicate that D_l and u can be obtained if k is known. The parameter k can be obtained in the following manner:

Using equations (1) and (20) (letting y = 0) and rearranging,

$$\frac{C_m t_m}{Ct} = \exp[\frac{(x-ut)^2}{4D_l t} - \frac{(x-ut_m)^2}{4D_l t_m}] \quad (27)$$

or

$$\ln\frac{C_m t_m}{Ct} = \frac{(t - t_m)(u^2tt_m - x^2)}{4D_l tt_m} \quad (28)$$

Substituting (22) into (28),

$$\ln\frac{C_m t_m}{Ct} = (\frac{t - t_m}{t})(\frac{u^2tt_m - x^2}{x^2 - u^2t_m^2}) \quad (29)$$

adding $(t - t_m)/t$ to both sides of the above equation and then rearranging the right-hand side, the following equation is obtained:

$$\ln\frac{C_m t_m}{Ct} + \frac{t - t_m}{t} = \frac{u^2t_m}{x^2 - u^2t_m^2} \frac{(t - t_m)^2}{t} \quad (30)$$

Equation (30) is the equation of a straight line of the form

$$Y = kX \quad (31)$$

where

$$\left.\begin{aligned} X &= (t - t_m)^2/t \\ Y &= \ln(C_m t_m/Ct) + (t - t_m)/t \end{aligned}\right\} \quad (32)$$

k is the slope of the straight line.

Procedures to estimate dispersive parameters using the LG method can be summarized as follows: read t_m and C_m from a C-t curve; calculate values of X and Y using equation (32); plot the transformed data and measure the slope k; calculate D_l and u using equations (25) and (26); then use data from an observation point (x', y') (not located on the x-axis) to obtain t_m'. The value of D_t can then be obtained from equation (18):

$$D_t = -y'^2D_l/(x'^2 - u^2t_m'^2 - 4D_l t_m') \quad (33)$$

It should be noted that the relationship involving C_m and C in equation (32) is independent of the units used. Therefore, if the measured data are conductivity or resistivity instead of concentration, it is not necessary to transform the data to concentration values as long as the values are directly proportional to concentration.

4.2 Laboratory Example

Two-dimensional dispersion experiments were conducted at China University of Geosciences in 1986, using salt water as a tracer [for details see Jiao et al. (1988)]. The electrical conductivity (G) and time (t) data from one of the experiments is shown in Table 2. The corresponding curve is shown in Figure 3. The value of t_m can be read as 159.8 (min.). The distance from the tracer injection point to the observation point is 80 cm. The G-t data is transformed to X-Y data using equation (32).

The ascending and descending portions of the G-t curve (Figure 3) correspond to the two straight lines in the X-Y plane (Figure 4). The points a, o, b in Figure 3 correspond to the points a', o', b' in Figure 4. Theoretically, if the movement of the tracer does not violate any of the assumptions behind development of equation (1), the two straight lines should coincide. For this example, deviation of the

Table 2. Observed Time-Conductivity Data and Transformed X, Y Data for LG Example

t(min.)	G(μΩ/cm)	X	Y
146.17	0.04	1.28	2.81
146.67	0.07	1.18	2.26
147.33	0.09	1.06	2.00
148.17	0.12	0.92	1.72
149.55	0.18	0.71	1.31
150.00	0.21	0.64	1.16
150.50	0.24	0.58	1.03
151.00	0.26	0.52	0.95
152.50	0.37	0.35	0.59
153.18	0.42	0.29	0.47
153.50	0.44	0.26	0.42
154.07	0.48	0.22	0.33
156.00	0.59	0.09	0.13
157.00	0.63	0.05	0.06
158.00	0.66	0.02	0.02
160.00	0.67	0.00	0.00
161.50	0.65	0.02	0.03
163.67	0.59	0.09	0.13
164.50	0.56	0.13	0.18
165.00	0.54	0.16	0.22
166.00	0.51	0.23	0.27
166.50	0.49	0.27	0.31
167.00	0.47	0.31	0.35
167.50	0.45	0.35	0.40
169.50	0.40	0.50	0.51
169.87	0.37	0.59	0.59
170.67	0.34	0.67	0.68
171.50	0.30	0.79	0.80
173.00	0.25	1.00	0.98
173.50	0.23	1.08	1.07
174.00	0.21	1.15	1.17
174.50	0.19	1.23	1.26
176.33	0.14	1.54	1.56
180.23	0.07	2.31	2.25
185.52	0.03	2.82	3.10
194.10	0.03	3.20	3.10

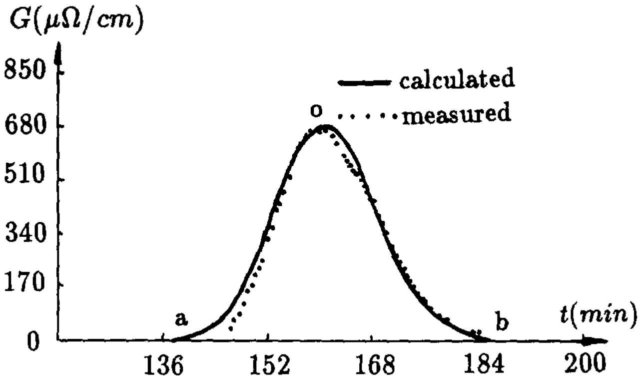

Fig. 3. Comparison of calculated and experimental conductivity changes as a function of time during a laboratory tracer dispersion study.

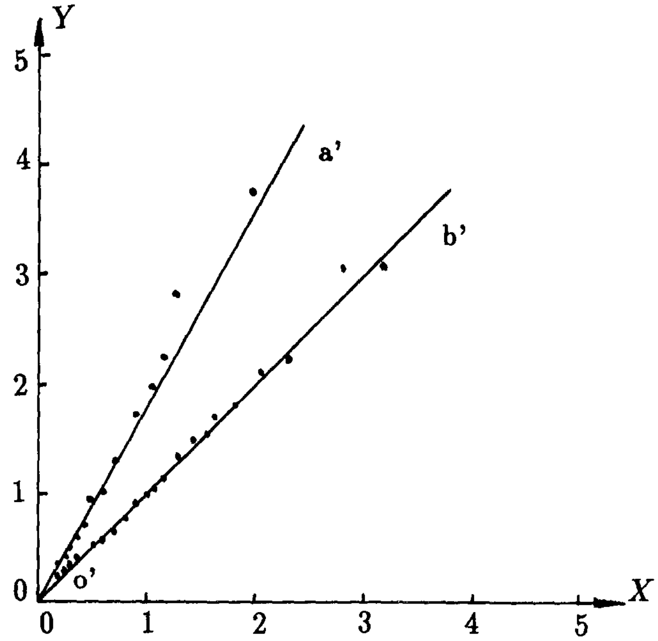

Fig. 4. X-Y line corresponding to Figure 3.

data points from the lines increases as X and Y increase. The deviation is an indication of the difference between the theoretical equation and the real situation. There are three possible explanations: (1) some of the assumptions for equation (1) may not be valid for this case; (2) the values t_m and C_m read from the C-t curve are not the exact values of t_m and C_m; or (3) the data are more erroneous at the time when the tracer first appears and at the time when it finally disappears at the observation point compared with the observed data near time t_m.

In this case, two slope values k_a (= 1.86) and k_b (= 1.0) are used. Employing equation (25) yields:

$$D_l = 80^2/[4 \times 159.8(159.8k_a + 1)] = 0.0335$$

Using equation (31):

$$u = 2(0.0335 \times k_a)^{1/2} = 0.4997$$

If k_b is used, then:

$$D_l' = 0.0622 \qquad u' = 0.4990$$

Averaged values of D_l and D_l', and u and u' are used as the final values $\bar{D}_l$ and $\bar{u}$

$$\bar{D}_l = (D_l + D_l')/2 = 0.0479(\text{cm}^2/\text{min.})$$

$$\bar{u} = (u + u')/2 = 0.4993(\text{cm/min.})$$

For this dispersion experiment, the observation points in the y direction were too far and did not intercept the tracer, so the data in the y direction were not available. D_t could not be obtained by using equation (33). Another method (Jiao et al. 1988) is used to obtain D_t to be 0.015 (cm^2/min.). Figure 4 shows that the fit between the real data and the calculated data is generally good.

5. Conclusion

Two methods of estimating dispersive parameters are described in the paper. The most interesting point in the DPA method is that for the first time it demonstrates that the aquifer dispersivities may be estimated by the shape of the dispersive plume, not necessarily the actual concentration. The example data in the text were collected with actual ground-water measurements, but the paper treats them in a way as if they were from resistivity measurements. So it may be reasonable to modify surface resistivity measurement methods to estimate aquifer dispersive parameters using the DPA method. This could be much more economical than using the traditional multiwell monitoring network to determine aquifer dispersive parameters. It may also be worthwhile to note that the DPA method does not necessarily need historical observation data (concentration-time data) which are often not available for field problems. In addition, it seems that this method can be used with data defining any plume resulting from instantaneous injection of a conservative material, even if the initial mass is not known. But it should be noted that the DPA method can only be used when the assumptions associated with equation (1) are valid, and the surface geophysical methods are applicable if the plume data are to be obtained through the surface measurements.

The LG method transforms the concentration-time curve into a straight line. Dispersivity and velocity can be estimated after measuring the slope of the line. This method is more economical and easier to perform compared to the method by Wang et al. (1987) which may suffer from the influence of erratic experiment data points on the derivative computation. On the other hand, the method in this paper is sensitive to the values of C_m and t_m. Attention should be paid to choose C_m and t_m properly.

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